Semigroups for Delay eq., Solutions 5

Ex 1: Fix, for the moment, some $t_0 > 0$.

Then any other t can be written as

$t = \lfloor t/t_0 \rfloor t_0 + \alpha_t$, $\lfloor t/t_0 \rfloor \in \mathbb{N}$ $\alpha_t \in [0, t_0)$.

Then do the following:

$$\frac{p(t)}{t} = \frac{p(\lfloor t/t_0 \rfloor t_0 + \alpha_t)}{\lfloor t/t_0 \rfloor t_0 + \alpha_t}$$

$$\overset{\text{sub. add.}}{\leq} \frac{\lfloor t/t_0 \rfloor p(t_0) + p(\alpha_t)}{\lfloor t/t_0 \rfloor t_0 + \alpha_t}$$

Then we see that as $t \to \infty$ also $\lfloor t/t_0 \rfloor \to \infty$.

Btw. we also do need to assume $\underline{p \text{ is bounded}}$ on compact sets! Then we see

$$\limsup_{t \to \infty} \frac{p(t)}{t} \leq \frac{p(t_0)}{t_0}$$

Now we take infimum over $t_0 > 0$ & conclude

$$\limsup \frac{p(t)}{t} \leq \inf_{t_0 > 0} \frac{p(t_0)}{t_0} \leq \liminf_{t \to \infty} \frac{p(t)}{t}$$

$\square$

Ex 2: Actually, I think the abscissa is $\omega_1(A)$, not $s(A)$:

$$\omega_1(A) = \inf\left\{ \mathrm{Re}\,\lambda : \exists \lim_{t\to\infty} \int_0^t ds\, e^{-\lambda s} T_s = R(\lambda, A) \right\}$$

Pf: Suppose λ is such that $\exists \lim_t \int_0^t e^{-\lambda s} T_s$ $(\Rightarrow \lambda \in \rho(A))$.
Then for any $x \in \mathcal{D}(A)$

$$x = (\lambda - A)^{-1}(\lambda - A)x = \lim_{t\to\infty} \int_0^t ds\, \underbrace{e^{-\lambda s} T_s (\lambda - A)x}_{-\frac{d}{ds}\left[e^{-\lambda s} T_s x\right]}$$

$$= - \lim_{t\to\infty} \left(e^{-\lambda t} T_t x - x \right)$$

$$\Rightarrow \exists \lim_{t\to\infty} e^{-\lambda t} T_t x = 0 \Rightarrow \underline{\omega_1(A) \le \mathrm{Re}\,\lambda}.$$

Next suppose $\mathrm{Re}\,\lambda > \omega_1(A)$. Then for any $x \in \mathcal{X}$.

$$e^{-\lambda t} T_t x - x = (A - \lambda) \int_0^t ds\, e^{-\lambda s} T_s x \quad \| \, (\lambda - A)^{-1}$$

$$\underbrace{e^{-\lambda t} T_t \underbrace{(\lambda - A)^{-1} x}_{\in \mathcal{D}(A)}} - (\lambda - A)^{-1} x = - \int_0^t ds\, e^{-\lambda s} T_s x$$

$\longrightarrow 0$ as $t \to \infty$ since $\mathrm{Re}\,\lambda > \omega_1(A)$.

$$\therefore \exists \lim_{t\to\infty} \int_0^t ds\, e^{-\lambda s} T_s x = (\lambda - A)^{-1} x$$

$\square$

Ex 3: First recall two implications

$$\mu > \omega(x,T) \Rightarrow e^{-\mu t}\|T(t)x\| \longrightarrow 0$$

and

$$e^{-\mu t}\|T(t)x\| \longrightarrow 0 \Longrightarrow \mu \geq \omega(x,T).$$

Now, just write, for any $\mu \in \mathbb{R}$

$$e^{-\mu t}\|T(t)x\| = e^{t\overbrace{\left(-\mu + \frac{1}{t}\log\|T_t x\|\right)}^{c_t}}, \quad (t \geq 0).$$

From this "representation" we see easily that if $\mu < \limsup \frac{1}{t}\log\|T(t)x\|$ then c_t is <u>not</u> eventually negative as $t \to \infty$, so

$$e^{-\mu t}\|T(t)x\| \not\longrightarrow 0, \quad t \to \infty \Longrightarrow \underline{\mu \leq \omega(x,T)}.$$

If, on the other hand,

$$\mu > \limsup \frac{1}{t}\log\|T(t)x\|,$$

then c_t <u>is</u> eventually $\leq -\varepsilon < 0$ for some $\varepsilon > 0$, so

$$e^{-\mu t}\|T(t)x\| \longrightarrow 0, \text{ as } t \to \infty \Longrightarrow \mu \geq \omega(x,T)$$

Thus we can deduce

$$\limsup_{t\to\infty} \frac{1}{t}\log\|T(t)x\| = \omega(x,T).$$

Ex 4. Call $[0,h] = I$ (actually any small interval $[0,\delta]$ would do).

Counterassumption: $\exists$ a sequence of points s_k such that $s_k \to \infty$ as $k \to \infty$ and for some $\varepsilon > 0$

$$F(s_k) \geq \varepsilon > 0 \;, \quad \forall k \in \mathbb{N} .$$

Let us also assume $|s_{k+1} - s_k| > \ell(I) = h$, $\forall k$, so that $(s_{k+1} - I) \cap (s_m - I) = \emptyset$, $k \neq n$.

Now, the hypothesis implies for all k that

$$F(s_k - t) \geq \frac{F(s_k)}{m} \geq \frac{\varepsilon}{m} \;, \quad (\forall t \in I),$$

so that

$$\int_{s_k - I} F(t)\, dt \geq \frac{\varepsilon\, \ell(I)}{m} \;, \quad \underline{\forall k \in \mathbb{N}} .$$

Therefore we reach a contradiction

$$\int_0^\infty F = \infty .$$

$\square$

Ex 5: A few facts to start with:

$$A_n = \sum_{k=1}^{n-1} |e_k\rangle\langle e_{k+1}|$$

$$\Rightarrow A_n^j = \sum_{k=1}^{n-j} |e_k\rangle\langle e_{k+j}|$$

$$\Rightarrow \underline{A_n^n = \bar{0}} \quad \Rightarrow \quad \sigma(A) = \{0\}$$ (which is clear anyways)

$\Rightarrow$ Important picture to bear in mind

$$e^{tA_n} = \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ 0 & 1 & t & \cdots & t^{n-2}/(n-2)! \\ \vdots & 0 & 1 & & \vdots \\ & & & \ddots & \\ & & & 1 & t \\ & & \cdots & 0 & 1 \end{bmatrix}$$

Some bounds: $\underline{\|A_n\| = 1} \Rightarrow \underline{\|e^{tA_n}\| \leq e^{t\|A_n\|}} = \underline{e^t}$.

$$\Rightarrow \|\underbrace{e^{i2\pi nt} \cdot e^{tA_n}}_{=:T_n(t)}\| = \|e^{tA_n}\| \leq e^t.$$

...Ex 5 We also recall a general fact: Let H_n be Hilbert

If $V_n \in \mathcal{B}(H_n)$ and $V := \bigoplus_{n=1}^{\infty} V_n$ is a bounded operator on $\bigoplus_{n=1}^{\infty} H_n$, then

$$\|V\| = \sup_n \|V_n\|.$$

Pf.

$$\|Vx\|^2 := \sum_{n=1}^{\infty} \underbrace{\|V_n x_n\|^2}_{\le \|V_n\| \|x_n\|} \le \sup_n \|V_n\|^2 \overbrace{\sum \|x_n\|^2}^{\|x\|^2}, \quad \left(\forall x \in \bigoplus_n H_n\right).$$

$\Rightarrow \|V\| \le \sup_n \|V_n\|$. On the other hand

$$\sup_n \|V_n\| \le \|V_{n_0}\| + \varepsilon \quad \text{for some } n_0$$

$$\le \|V_{n_0} x_{n_0}\| + 2\varepsilon \quad \text{for some } x_{n_0} \in H_{n_0}.$$

Thus, by extending x_{n_0} by zero to an element of $\bigoplus_n H_n$, we get the other inequality (for arbit. $\varepsilon > 0$)

$$\sup_n \|V_n\| \le \|V\| + 2\varepsilon. \quad \text{Done.}$$

Now, define $T(t) := \bigoplus_{n=1}^{\infty} T_n(t)$,

where $T_n(t) = e^{i2\pi n t} \cdot e^{tA_n}$ is a unif. cont. semig. on $\mathbb{C}^n$.

We see, by above,

$$\|T(t)\| = \sup_n \|T_n(t)\| \le e^t \quad (t \ge 0)$$

$\Rightarrow$ $\underline{\omega_0(T) \le 1}$ (As soon as we show T is a s.g.)

... Ex 5 T is C_0-s.g. :

Properties $T_0 = \mathbb{1}$, $T_{t+s} = T_t T_s$ are clear.

Strong cont. follows from the fact that T is clearly strongly cont. on the dense set of "eventually zero sequences of $\bigoplus_{n=1}^{\infty} H_n$". Done.

Generator A is $\underline{\bigoplus_{n=1}^{\infty} (i2\pi n + A_n) =: \tilde{A}}$ with the domain $\underline{\mathcal{D}(\tilde{A}) = \{ x : \sum_{n=1}^{\infty} n^2 \|x_n\|^2 < \infty \}}$:

Pick $x \in \mathcal{D}(A)$. Then it's easy to check that $(Ax)_n$ must, not surprisingly, be $(i2\pi n + A_n)x_n$.

Then, since $Ax \in \bigoplus H_n$, we have

$$\infty > \|Ax\|^2 = \sum_{n=1}^{\infty} \underbrace{\|i2\pi n x_n + A_n x_n\|^2}_{(2\pi n)^2 \|x_n\|^2 + \|A_n x_n\|^2}$$

$$\geq (2\pi)^2 \sum_{n=1}^{\infty} \|x_n\|^2 n^2 .$$

Thus $x \in \mathcal{D}(\tilde{A})$. Therefore we have shown A is a part of $\tilde{A}$, $A \subseteq \tilde{A}$.

We could go and do the inclusion $\tilde{A} \subseteq A$ directly. However, since A IS the generator of a "$\omega_0(T) \leq 1$ - s.g.", we know that for $\lambda > 1$ $\mathcal{R}(\lambda - A) = \bigoplus_n H_n$. Therefore, if we can show $\exists (\lambda - \tilde{A})^{-1}$ too for some $\lambda > 1$,

... Exc 5: ... we see then that $\tilde{A}$ cannot be a proper extension of A, because this would imply $\lambda - \tilde{A}$ is a proper injective ext. of the full-range-operator $\lambda - A$.

Therefore we prove:

$\exists\, R(\lambda, \tilde{A})^{-1}$ for all $\mathrm{Re}\,\lambda > 0$

Let $\mathrm{Re}\,\lambda > 0$. First of all it's easy to see each $i2\pi n + A_n$ has the resolvent

$$(\lambda - i2\pi n - A_n)^{-1} = \sum_{k=0}^{n-1} \frac{A_n^k}{(\lambda - i2\pi n)^k},$$

since $A_n^n = 0$. For n, such that $|\lambda - i2\pi n| > 1$ this gives a bound

$$\|R(\lambda, i2\pi n + A_n)\| \leq \frac{1}{|\lambda - i2\pi n| - 1}$$

Since there is at most one n with $|\lambda - i2\pi n| \leq 1$ we still get the bound

$$\sup_n \|R(\lambda, i2\pi n + A_n)\| < \infty.$$

Therefore, if we define $R(\lambda, \tilde{A}) := \bigoplus_{n=1}^{\infty} R(\lambda, i2\pi n + A_n)$, $R(\lambda, \tilde{A})$ is in $\mathcal{B}(\bigoplus_n H_n)$ and clearly the inverse of $\lambda - \tilde{A}$. Done.

... Ex 5 : Consequences : $A = \bar{A}$ and $\underline{S(A) \leq 0}$.

But it's also clear that $i2\pi n \in \sigma(A)$, $\forall n \in \mathbb{N}$

so in fact $\underline{S(A) = 0}$.

$\underline{\omega_0(T) \geq 1}$ (so that $\omega_0(T) = 1$) :

Fix $\mu < 1$. It is sufficient to find elements x^m such that $\|x^m\| = 1$ and

$e^{-\mu m} \| T(m) x^m \| \geq \varepsilon > 0$, $(\forall m \in \mathbb{N})$.

Note first that $\underline{T(t) = (e^{tA_n})_n}$.

Let's look at the elements of the form

$x_n := n^{-1/2} (1, \ldots, 1) \in \mathbb{C}^n$.

Then $\|x_n\| = 1$, $A_n x_n = n^{-1/2}(1, \ldots, 1, 0)$ so x_n is "almost an eigenvector as n is big".

We might from this guess $e^{tA_n} x_n$ is not "far from $e^t x_n$".

Indeed

$$\| e^{tA_n} x_n - e^t x_n \| = \| \int_0^t ds\, e^{sA_n} e^{t-s} (A_n x_n - x_n) \|$$

$$\leq \underbrace{\|A_n x_n - x_n\|}_{\leq n^{-1/2}} \int_0^t \underbrace{\|e^{sA_n}\|}_{\leq e^s} e^{t-s}$$

$$\underline{\leq t e^t n^{-1/2}}.$$

... <u>Ex 5</u>: This implies

$$\|e^{tA_n}x_n\| \geq e^t - te^t n^{-1/2} = e^t(1 - t/\sqrt{n}).$$

Now let x^n be "extension of x_n by zero". Then

$$e^{-\mu m}\|T(m)x^n\| \geq e^{-\mu m}\|e^{mA}x^n\|$$

$$\geq e^{m(1-\mu)}(1 - m/\sqrt{n})$$

So by coupling $n = 4m^2$, then $1 - m/\sqrt{n} = \frac{1}{2}$ and

$$e^{-\mu m}\|T(m)x^{4m^2}\| \geq e^{m(1-\mu)}\frac{1}{2} \to \infty \text{ as } m \to \infty. \text{ Done.}$$

<u>$\omega_1(A) = 1$</u>

We elaborate the idea above a bit. Let $x_n \in \mathbb{C}^n$ be defined as before. Now let $y_n := \varepsilon_n x_n$, where $\varepsilon_n \geq 0$ are to be determined, but so that y still belongs to $D(A)$:

$$\sum_n \varepsilon_n^2 n^2 < \infty.$$

Let's look at the estimate for any n

$$e^{-\mu m}\|T(m)y\| \geq e^{-\mu m}\|e^{mA_n}\underbrace{y_n}_{\varepsilon_n x_n}\|$$

$$\geq \varepsilon_n e^{m(1-\mu)}(1 - m/\sqrt{n}) \quad \text{as above.}$$

As before, put $n = 4m^2$ so that the last line becomes $\varepsilon_{4m^2} e^{m(1-\mu)}\frac{1}{2}$.

... **Ex 5** ... It is now easy to bound this from below by putting $\varepsilon_{4m^2} := e^{-m(1-\mu)}$, or more precisely

$\varepsilon_k = 0$, $\sqrt{k}/2 \notin \mathbb{N}$ and $\varepsilon_k = e^{-\frac{\sqrt{k}}{2}(1-\mu)}$, $\sqrt{k}/2 \in \mathbb{N}$.

Finally, since $\varepsilon_k \to 0$ so fast as $k \to \infty$ that

$$\sum_{n=1}^{\infty} \varepsilon_n n^2 = \sum_{m=1}^{\infty} \varepsilon_{4m^2} (4m^2)^2 = \sum_{m} e^{-m(1-\mu)} 4^2 m^4 < \infty,$$

the condition $y \in D(A)$ is satisfied. $\square$