Semigroups & Delay eq, Solutions 4.

Ex 1: We only need to recall that

$$T(t)^{\odot} = T^*(t)\big|_{\mathcal{X}^{\odot}}, \text{ and that}$$

$$R(\lambda - A^*)^{-1} = D(A^*) \subseteq \mathcal{X}^{\odot}.$$

(Why? Since for $x^* \in D(A^*)$ we have

$$\|T^*(t)x^* - x^*\| = \left\| \int_0^t T^*(s) A^* x^* \right\|$$

$$\leq \int_0^t M e^{s\omega_0} \|A^* x^*\| \, ds$$

$$\longrightarrow 0 \quad \text{as } t \to 0. \quad)$$

Now then, for $\lambda \in \rho(A^*)$,

$$(\lambda - A^*) T(t)^{\odot} (\lambda - A^*)^{-1} = (\lambda - A^*) \underbrace{T^*(t)\big|_{\mathcal{X}^{\odot}} (\lambda - A^*)^{-1}}_{T^*(t)(\lambda - A^*)^{-1}}$$

$$= \underbrace{(\lambda - A^*)(\lambda - A^*)^{-1}}_{1} T^*(t),$$

since the semigroup T^* and its generator A^* commute.

Ex 2.: We will need the general fact that the adjoint T^* of any bounded $T \in \mathcal{B}(\mathcal{X})$ is $\sigma(\mathcal{X}^*,\mathcal{X})$-$\sigma(\mathcal{X}^*,\mathcal{X})$ cont. mapping $\mathcal{X}^* \to \mathcal{X}^*$ (This was also done in the last problem set). For, for any $x \in \mathcal{X}$ the mapping

$$x^* \longmapsto \langle x, T^* x^* \rangle = \langle \underbrace{Tx}_{\in \mathcal{X}}, x^* \rangle$$

is $\sigma(\mathcal{X}^*,\mathcal{X})$ cont. Done.

By the preceding result, and weak*-cont. of $f: [a,b] \to \mathcal{X}^*$, we know $s \mapsto T^* f(s)$ is also weak*-cont.

Finally, one just manipulate

$$\begin{aligned}
\langle x, T^* \int_a^b f(s) \rangle &= \langle Tx, \int_a^b f(s) \rangle \\
&= \int_a^b \underbrace{\langle Tx, f(s) \rangle}_{\langle x, T^* f(s) \rangle} \\
&= \langle x, \int_a^b T^* f(s) \rangle, \quad (x \in \mathcal{X}). \quad \square
\end{aligned}$$

weak* cont. so the <u>integral is well def.</u> by the last exercise.

If $S \in \mathcal{B}(\mathcal{X}^*)$ is not adjoint operator, then the above manipulation can fail in two ways:

1° $S^* x \in \mathcal{X}^{**}$ need not be in $\mathcal{X}$.

2° $s \mapsto S f(s)$ need not be weak*-cont, so the <u>integral is ill-defined</u>.

Ex 3: The most fun way to see this, I think, is to note we can write, for(+) $\lambda \neq 0$,

$$\mathbb{1} = \frac{1}{\lambda}\lambda = \frac{1}{\lambda}(\lambda - A + A) \qquad \underline{\text{on } D(A)}.$$

We simply need to apply this to both sides of the resolvent $(\lambda - A)^{-1}$:

To the left:

$$\underbrace{(\lambda - A)^{-1}x}_{\in D(A)} = \frac{1}{\lambda}(\lambda - A + A)(\lambda - A)^{-1}x$$

$$= \frac{1}{\lambda}(\mathbb{1} + A(\lambda - A)^{-1})x \qquad (\underline{\forall x \in X}).$$

To the right:

$$(\lambda - A)^{-1}x = (\lambda - A)^{-1}\frac{1}{\lambda}(\lambda - A + A)x$$

$$= \frac{1}{\lambda}(\mathbb{1} + (\lambda - A)^{-1}A)x, \qquad (\underline{\underline{\forall x \in D(A)}}).$$

So, for all $x \in D(A)$ we see, by comparison,

$$A(\lambda - A)^{-1}x = (\lambda - A)^{-1}Ax \qquad \square$$

(+) Oh yes, and if $\lambda = 0$ then the claim is trivial of course.

Ex 4: Suppose $\lambda_0 \in \rho(T)$, $T \in \mathcal{B}(\mathfrak{X})$. Then

$$\lambda - T = (\lambda - \lambda_0) + (\lambda_0 - T)$$
$$= (\lambda_0 - T)^{-1}\left[(\lambda - \lambda_0)(\lambda_0 - T)^{-1} + \mathbb{1}\right],$$

so that as soon as $\|(\lambda-\lambda_0)(\lambda_0-T)^{-1}\| < 1$ the inverse of the term in brackets can be defined by geometric series, implying that $\exists\,(\lambda - T)^{-1}$ and $\lambda \in \rho(T)$.

$\therefore$ $\rho(T)$ is open, and $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is closed.

(Alternatively, one can note $\Phi : \mathbb{C} \to \mathcal{B}(\mathfrak{X})$, $\Phi(\lambda) := \lambda - T$ is cont. so that $\rho(T) = \Phi^{-1}(G(\mathcal{B}(\mathfrak{X})))$ is open, since the group $G(\mathcal{B}(\mathfrak{X}))$ of invertible elements is open in Banach algebra.)

Finally, if $|\lambda| > \|T\|$, we see by the same argument that

$$\lambda - T = \lambda(\mathbb{1} - T/\lambda)$$

has an inverse (since $\|\frac{T}{\lambda}\| < 1$).

Thus $\sigma(T)$ must be a bounded set (contained in $\overline{B}(0, \|T\|)$) and, since it's closed, compact.

<u>Non-emptiness</u>: If $\sigma(T) = \emptyset$ then $R(\lambda, T)$ would be an entire function. But it would also be bounded, for
$1 = \|\mathbb{1}\| = \|(\lambda - T)R(\lambda, T)\| \geq (|\lambda| - \|T\|)\|R(\lambda, T)\|$ for all $|\lambda| > \|T\|$.
By Liouville, $\lambda \mapsto R(\lambda, T)$ would be constant $\bar{0}$ RR □

Ex 5: Let's recall the definition:

$T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, ($\mathfrak{X}, \mathfrak{Y}$ Banach spaces) is compact if $\overline{T(U)}$ is compact, where U is the unit ball in $\mathfrak{X}$.

Now, since in metric spaces compactness and sequential compactness coincide, it is sufficiently obvious that we have

$\overline{T(U)}$ is compact

$\Leftrightarrow$ $\overline{T(U)}$ is seq. compact

$\Leftrightarrow$ Every seq. in $\overline{T(U)}$ has a cluster point in $\overline{T(U)}$

$\Leftrightarrow$ Every seq. in $T(U)$ ——— " ——— in $\overline{T(U)}$

$\Leftrightarrow$ Every seq $(Tx_n)_{n \in \mathbb{N}}$, $x_n \in U$ has a cluster point (in $\overline{T(U)}$).

$\Leftrightarrow$ (iii)

$\therefore$ (i) $\Leftrightarrow$ (iii) which was pretty clear, wasn't it?

Next the interesting part (ii) $\Leftrightarrow$ (iii):

The crux of it is this:

Lemma

Metric space E is totally bounded if and only if every seq. $(x_n)_n \subseteq E$ has a Cauchy sub. seq.

(Note the analogy to sequential compactness!)

... Ex 5: Proof. Suppose E is not totally bounded. Then, by def., $\exists\, \varepsilon > 0$ s.t. E cannot be covered by finitely many balls of radius ε.

We construct a seq. with no Cauchy sub.seq:
Pick any $x_1 \in E$. Then $B(x_1, \varepsilon)$ cannot cover E.
Thus we can pick next $x_2 \in E \setminus B(x_1, \varepsilon)$, and still $B(x_2, \varepsilon) \cup B(x_1, \varepsilon)$ cannot cover E.
Continuing in obvious manner we are able to find a seq. $(x_n)_n$ s.t. $d(x_n, x_k) > \varepsilon$, $\forall n, k \in \mathbb{N}$.
Such $(x_n)_n$ cannot have Cauchy sub.seq. Done.

Next assume E be totally b, and pick a sequence (x_n). If $\{x_n\}_n \subseteq E$ was finite then some x_m would have to be "repeated" ∞-many times, so surely there would be a Cauchy sub.seq ("constant x_m").
So, we may assume $\{x_n\} \subset E$ is infinite.

First cover E by finitely many balls of rad. 2^{-1}.
One of the balls, B_1 say, will contain ∞-many x_n's.
Define x_{n_1} to be the first x_n in B_1.

Next cover B_1 by finitely many balls of rad 2^{-2}.
At least one of them, B_2 say, contains ∞-many $x_n \in B_1$.
Define x_{n_2} to be the first $x_n \in B_2 \cap B_1$ s.t. $n > n_1$.

... Ex 5: Iterating this process gives us in the end a subseq. such that $x_{n_k} \in B_m$, $\forall k \geq m$, implying it's Cauchy (Radius of B_m is 2^{-m}). Q.E.D.

To prove (ii) <=> (iii) let $E = T(U)$ and add the fact that the "background space" Y is complete, so that each Cauchy seq. converges, and we have:

Corollary:

$T(U) \subseteq Y$ (Y Banach) is totally bounded iff every seq. $\{Tx_n\} \subseteq T(U)$ has a converging sub seq.

That, I think, is as close an equivalence as we need here. □