Semigroups & Delay eq., Solutions 2.

Ex 1: By continuity of the mappings $t \mapsto f(t)$ and $t \mapsto Af(t)$ the integrals can be taken in the sense of Riemann. By definition, then, we have the existence of

$$\lim_{N\to\infty} \sum_{k=1}^{N} Af(t_k)\Delta t_k = \lim_{N} A\left(\sum^{N} f(t_k)\Delta t_k\right).$$

But now observe that we also have the existence

$$\exists \lim_{N\to\infty} \sum^{N} f(t_k)\Delta t_k = \int_a^b f(t)\,dt,$$

so that, by the very <u>definition of closedness</u>, we can deduce that $\underline{\int_a^b f \in D(A)}$ and

$$A\left(\int_a^b f(t)\,dt\right) = \int_a^b Af(t)\,dt. \quad \square$$

<u>Alternative way</u>: We can assume $D(A)$ is dense; otherwise take $X = \overline{D(A)}$. Then, since A is also <u>closed</u>, we know the adjoint A^* is <u>densely</u> defined.

For every $x^* \in D(A^*)$

$$\langle A\int_a^b f(t), x^*\rangle = \langle \int_a^b f(t), A^*x^*\rangle = \int_a^b \langle f(t), A^*x^*\rangle$$

$$= \int_a^b \langle Af(t), x^*\rangle = \langle \int_a^b Af(t), x^*\rangle.$$

Okay, I forgot to check $\int_a^b f \in D(A)$, but in our "semigroup-case" it's known. If you accept that, then the above implies $A\int_a^b f = \int_a^b Af$ by denseness of $D(A^*)$. "Done".

Ex 2: (i) Since $\mathcal{Y}$ is complete, it's sufficient to show $\{A_n x\}_n$ is Cauchy for all $x \in \mathcal{X}$:

Given $\varepsilon > 0$, choose $y \in E$ such that $\|x - y\| < \varepsilon$. Then estimate

$$\|A_n x - A_k x\| \leq \|A_n(x-y)\| + \|A_n y - A_k y\| + \|A_k(y-x)\|$$

$$\leq 2M\|x-y\| + \|A_n y - A_k y\|,$$

by uniform boundedness of A_n. Now, since $\{A_n y\}_n$ is Cauchy, we find $N_{\varepsilon,y}$ such that for $n, k \geq N_{\varepsilon,y}$ $\|A_n y - A_k y\| \leq \varepsilon$. This finally gives

$$\|A_n x - A_k x\| \leq (2M+1)\varepsilon, \qquad (n, k \geq N_{\varepsilon,y}). \quad \square$$

(ii) It clear such extension A is linear. Boundedness follows from

$$\|\lim_n A_n x\| = \lim_n \|A_n x\| \leq M\|x\|, \quad (x \in \mathcal{X}).$$

Remark: Completeness of only $\mathcal{Y}$ is required.

Ex 3 : (ii) $\Rightarrow$ (i) :

Suppose $\mathcal{R}(\lambda_0 - A) = \mathcal{X}$ for some $\lambda_0 > 0$.

Now, given $\lambda > 0$ and any $y \in \mathcal{X}$, let's see if we can find $x \in \mathcal{D}(A)$ such that

$$(\lambda - A)x = y.$$

It feels like a good idea to add and subtract λ_0:

$$(\lambda - \lambda_0)x + (\lambda_0 - A)x = y.$$

We can apply $(\lambda_0 - A)^{-1}$ to both sides resulting

$$[\mathbb{1} - (\lambda_0 - \lambda)(\lambda_0 - A)^{-1}]x = (\lambda_0 - A)^{-1}y.$$

But $(\lambda_0 - \lambda)(\lambda_0 - A)^{-1}$ is a bounded operator of norm less than or eq. to $|\lambda_0 - \lambda| \frac{1}{\lambda_0}$ (by the dissipativity).

Therefore if $|\lambda_0 - \lambda| < \lambda_0$, we can define the inverse $[\mathbb{1} - (\lambda_0 - \lambda)(\lambda_0 - A)^{-1}]^{-1}$ by the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{\lambda_0 - \lambda}{\lambda_0 - A}\right)^n.$$

Thus, finally,

$$x = \sum_{0}^{\infty} \frac{(\lambda_0 - \lambda)^n}{(\lambda_0 - A)^{n+1}} y,$$

and $\mathcal{R}(\lambda - A) = \mathcal{X}$ for all $\lambda \in (0, 2\lambda_0)$. Iteration then gives the same for all $\lambda \in (0, \infty)$. $\square$

Ex 4: That $D(A) = \{f \in C[0,1] : f(0) = 0\}$ is not dense in $(C^1[0,1], \|\cdot\|_\infty)$ is clear, since for any $f \in C^1[0,1]$ such that $f(0) \neq 0$ we see that its neighborhood $\{g \in C^1 : \|g-f\|_\infty < |f(0)|\}$ is disjoint from $D(A)$.

Now, given $\lambda > 0$ and $f \in D(A)$ we must prove

$$\|\lambda f - Af\|_\infty = \|\lambda f + f'\|_\infty \geq \lambda \|f\|_\infty .$$

The crucial observation is that $|f|$ indeed <u>attains its maximum</u> $\|f\|_\infty$. That is, there exists $x_0 \in [0,1]$ such that $|f(x_0)| = \|f\|_\infty$.

The next thing is to note that $f'(x_0)$ must be zero <u>unless</u> $x_0 = 1$, in which case it must, by maximality of $|f(1)|$, still be true that $|\lambda f(1) + f'(1)| \geq \lambda |f(1)|$. (You can assume, by rotation, that $f(1) > 0$; then it should be apparent...)

So, in any case we have

$$\begin{aligned}\|(\lambda + A)f\|_\infty &\geq |\lambda f(x_0) + f'(x_0)| \\ &\geq \lambda |f(x_0)| \\ &= \lambda \|f\|_\infty . \quad \square\end{aligned}$$

Ex 5 : Let $\mathcal{X} = BUC(\mathbb{R})$ or $L^p(\mathbb{R})$

and look at the differential eq. for some given $g \in \mathcal{X}$.

$$\lambda f + f' = g \quad \Big\| \, e^{\lambda x}$$

$$\Leftrightarrow \frac{d}{dx}\left[e^{\lambda x} f(x)\right] = e^{\lambda x} g(x) \quad \Big\| \, \text{"} \int_a^x \text{"}$$

$$\Leftrightarrow e^{\lambda x} f(x) - e^{\lambda a} f(a) = \int_a^x e^{\lambda y} g(y) \, , \quad (x, a \in \mathbb{R}).$$

This much we can always do. First question is what to with the term $e^{\lambda a} f(a)$? If we can somehow "handle it", the second question is whether resulting f belongs to $\mathcal{X}$.

If either $\underline{\operatorname{Re}\lambda > 0}$ or $\underline{\operatorname{Re}\lambda < 0}$, we can let $a \to -\infty$ or ∞ respectively, thus obtaining

$$f(x) = \underbrace{\int_{-\infty}^x e^{-\lambda(x-y)} g(y)}_{\operatorname{Re}\lambda > 0} \, , \quad \text{or} \quad -\int_x^\infty e^{-\lambda(x-y)} g(y) \, , \quad \operatorname{Re}\lambda < 0 .$$

$$= (\Theta(y) e^{-\lambda y}) * g\,(x) \, , \quad \text{or} \quad -(\tilde{\Theta}(y) e^{-\lambda y}) * g\,(x)$$

where $\Theta = \chi_{[0,\infty)}$ and $\hat{\Theta} = \chi_{(-\infty,0]}$. Now, it is a general fact that for all $p \in [1,\infty]$

$$\|\psi * \varphi\|_p \le \|\psi\|_1 \|\varphi\|_p \, , \quad (\psi \in L^1, \varphi \in L^p).$$

(Proof by Jensen: Assuming $\overset{p \neq \infty}{\psi \ge 0}$ $\|\psi\|_1 = 1$

$$\int dx \left| \int dy\, \psi(y) \varphi(x-y) \right|^p \le \int dx \int dy\, \psi(y) |\varphi(x-y)|^p = \|\varphi\|_p^p \; \square \,)$$

... **Ex 5**: Applying it to our case with

$\varphi = \theta e^{-\lambda y}$ or $\tilde{\theta} e^{-\lambda y}$, so that $\|\varphi\|_1 = \frac{1}{|\mathrm{Re}\,\lambda|}$,

we get the bound

$\|f\|_p \le \frac{1}{|\mathrm{Re}\,\lambda|} \|g\|_p$.

Thus it should be clear that in all cases of $\mathfrak{X}$ we have $R(\lambda - A) = \mathfrak{X}$ and $(\lambda - A)^{-1}$ is bounded. Since this holds true for all $\mathrm{Re}\,\lambda \neq 0$ we have $\sigma(A) \subseteq i\mathbb{R}$.

Case $\mathfrak{X} = BUC(\mathbb{R})$, $\lambda \in i\mathbb{R}$:

In this case $\lambda \in P\sigma(A_{BUC})$ for

$\lambda f + f' = 0 \iff f(x) = f(0) e^{-\lambda x} \in BUC(\mathbb{R})$,

for arbitrary $f(0) \in \mathbb{C}$.

The same reasoning shows that $\lambda f + f' = 0$ has no solutions when $\mathfrak{X} = L^p$, for $e^{-\lambda x} \notin L^p$, $p < \infty$.

Case $\mathfrak{X} = L^p(\mathbb{R})$, $\lambda \in i\mathbb{R}$:

Suppose $g \in R(\lambda - A)$, so that $\exists f \in D(A)$ such that

$\lambda f + f' = g$.

As before, we have that for all x, a holds

... Ex 5: $f(x)e^{\lambda x} - f(a)e^{\lambda a} = \int_a^x e^{\lambda y} g(y)\,dy$.

Now, since $f \in D(A)$, f is cont. and vanishes at $\pm\infty$. This implies g must satisfy

$$\exists \lim_{\substack{x \to \infty \\ a \to -\infty}} \int_a^x e^{\lambda y} g(y)\,dy \text{ ''} = \hat{g}(i\lambda) \text{''} \underline{\underline{= 0}}.$$

Subcase $p = 1$:

When $g \in L^1$ the $\hat{g}$ is actually pointwise well-def. In fact we have $\|\hat{g}\|_\infty \le \|g\|_1$, so that for fixed λ

$L^1 \ni g \longmapsto \hat{g}(i\lambda)$

is cont. Hence $\{g \in L^1 : \hat{g}(i\lambda) = 0\}$ is a closed proper subspace which contains $R(\lambda - A)$.

$$\therefore \; i\mathbb{R} = R\sigma(A|_{L^1}).$$

Subcase $p \in (1, \infty)$:

We show that actually $\overline{R(\lambda - A)} \supseteq L^p_c$ ("compactly supported L^p"), so that it is dense.

Let $g \in L^p_c$ such that $\operatorname{spt} g \subseteq [\alpha, \beta]$, and

$$\int_{-\infty}^{\infty} e^{\lambda y} g(y) = \int_\alpha^\beta e^{\lambda y} g(y) =: A \in \mathbb{C}.$$

Note that when $A \neq 0$ g does not satisfy the necessary cond. to be in the range of $\lambda - A$.

... Ex 5 On the other hand, if A is zero then $x \longmapsto \int_{-\infty}^{x} e^{\lambda y} g$ is compactly supported and thus in L^p too, so that $g \in R(\lambda - A)$.

Our job, then, is to "modify" g a tiny bit with respect to L^p-norm such that $A = 0$.

Define, for $\varepsilon > 0$,

$$\varphi_\varepsilon := -\varepsilon \cdot e^{\arg A} \chi_{[0, |A|/\varepsilon]} \in L^p_c .$$

Then, if $M = \max[|A|/\varepsilon, \beta]$,

$$\int_{-\infty}^{M} (e^{\lambda y} g(y) + \varphi_\varepsilon) = \int_{-\infty}^{M} e^{\lambda y} g + \int_{-\infty}^{M} \varphi_\varepsilon = 0$$

This, by the reasoning above, implies $g + e^{-\lambda y}\varphi_\varepsilon \in R(\lambda - A)$.

Finally

$$\| g - (g + e^{\lambda y}\varphi_\varepsilon)\|_p^p = \|\varphi_\varepsilon\|_p^p = \varepsilon^{p-1} A ,$$

which, because $p > 1$, can be made as small as we please. Done

$\therefore i\mathbb{R} = C\sigma(A_{L^p})$.

Finally, either by Hille-Yosida (since $\|(\lambda - A)^{-1} g\| \le \frac{\|g\|}{\lambda}, \lambda > 0$) or by Lumer-Phillips (since A is m-dissipative) A generates a strongly cont. contraction semigroup. $\square$