Semigroups & Delay equations, Solutions 1.

Ex 1. a) If $\lambda \in \rho(A)$ in the sense of Def. 2. then clearly $\lambda \in \rho(A)$ in the sense of Def. 1.

Assume then that $\lambda \in \rho(A)$ in the sense of Def. 1, and that A is <u>closed</u>, meaning $\mathcal{G}(A) \subset \mathcal{X} \times \mathcal{X}$ is closed. Then we see from the following easy equivalences

$$\mathcal{G}(A) \text{ is closed} \Leftrightarrow \mathcal{G}(\lambda - A) \text{ is closed}$$
$$\Leftrightarrow \mathcal{G}(\lambda - A)^{-1} \text{ is closed}$$

that $\mathcal{G}(\lambda-A)^{-1}$ is indeed closed. But since $(\lambda - A)^{-1}$ is <u>bounded</u> (by (iii)) this can only be if its domain $D(\lambda-A)^{-1} = R(\lambda - A)$ is closed.

$$\therefore\ \underline{R(\lambda - A) = \overline{R(\lambda-A)} \overset{(ii)}{=} \mathcal{X}}\ .$$

To find A for which $\rho(A) \neq \emptyset$ in the sense of Def. 1 but $\emptyset$ — " — Def. 2., we should start with a closed operator and restrict its domain in a suitable way. For example we could take generator "$\frac{d}{dx}$" in Ex 2. and set $A := \frac{d}{dx}\big|_{\mathcal{S}}$, where $\mathcal{S}$ is the Schwartz-class. Then, as you can check,

$$\mathcal{S} \subset R(\lambda - A) \neq L^p \text{ nor } C[-r,0]$$

for all $\lambda \in \mathbb{C}$.

Ex 1 b) "Computation" in a) already proved

$\rho(A) \neq \emptyset$ in the sense 2. $\Rightarrow$ A is closed.

For a closed A such that $\rho(A) = \emptyset$ consider multiplication operator $M_f : D \to L^p$

$$M_f(\varphi)(x) := f(x)\,\varphi(x) \quad (x \in \mathbb{R})$$

where $D = \{\varphi \in L^p : f\varphi \in L^p\}$, and f a measurable function. It is rely easy to see <u>M_f is closed</u> (check it!).

Finally, $\psi \in R(\lambda - M_f) \Leftrightarrow \exists \varphi \in D : \lambda\varphi - f\varphi = \psi$

$$\Leftrightarrow \text{"} \varphi = \frac{\psi}{\lambda - f} \in L^p \text{"}.$$

Therefore $R(\lambda - M_f) = L^p$ iff $\frac{\psi}{\lambda - f} \in L^p$ for all $\psi \in L^p$, which holds iff $\lambda \notin$ essential-range of f.

So, pick f measurable such that $\overline{f(\mathbb{R})} = \mathbb{C}$, and let $A := M_f$. Then

$\rho(A) = \emptyset$

Ex 2 The properties $T(0)=\mathbb{1}$ and $T(t+s)=T(t)T(s)$ are plain.

For the strong continuity it is sufficient to prove the limit $\lim_{t\downarrow 0}\|T(t)x-x\|=0$ for all x in a dense subset. (General fact; follows from the equicont. of the family $\{T(t): t\in[0,T]\}$)

As a dense subset in both $C_c[-r,0)$ and $L^p[-r,0]$ we pick $\mathcal{D}_c^\infty[-r,0)$. Then it's easy to check

$$\lim_{t\downarrow 0}\|T(t)\varphi(\theta)-\varphi(\theta)\|_{\mathcal{X}}=\lim_{t\downarrow 0}\|\varphi(\theta+t)-\varphi(\theta)\|_{\mathcal{X}}$$

$$=0 \qquad (\varphi\in\mathcal{D}_c^\infty[-r,0))$$

where $\mathcal{X}$ can be either $C_c[-r,0)$ or $L^p[-r,0]$.

The Generator A:

By the basic properties we know $f\in\mathcal{D}(A)$ iff there exists an element $g\in\mathcal{X}$ such that, for $t\geq 0$,

$$T(t)f(\theta)-f(\theta)=\int_0^t T(s)\,g(\theta)\,ds \qquad (\theta\in[-r,0)).$$

Then g is then Af. (Why? Since then $\exists\lim_{t\downarrow 0}\frac{T(t)f-f}{t}=g$)

All we have to do now is to see what the identity above means explicitly. In fact, fix θ to be $-r$. It becomes

$$f(-r+t)-f(-r)=\int_0^t g(-r+s)\,ds^{(\dagger)} \quad (\forall t\in[0,\infty)),$$

This is equivalent to f $\overbrace{\text{being absolutely cont.}}^{\text{considered as extended}}$ on $[-r,\infty)$ and $f'=g$

$(\dagger)$ g is extended by zero to $(0,\infty)$.

... **Ex 2**: Therefore, in both cases, $\mathcal{X} = L^p$ or C_0, we find that A "acts" like $\frac{d}{d\theta}$. The difference comes from the requirement that $g = Af$ must belong to $\mathcal{X}$.

Case 1. $C_0[-r,0)$: $\mathcal{D}(A_1) = \{ f \in C_0[-r,0) : f' \in C_0[-r,0) \}$

Case 2. $L^q[-r,0]$: $\mathcal{D}(A_2) = [f \in L^q[-r,0] : f' \in L^q[-r,0], \underline{f(0)=0}]$

The $f(0)=0$ follows from the subtle fact that the extension of f to $[-r,\infty)$ must have the representation $(\dagger)$.

"Resolvent $(\lambda - A)^{-1} = \int_0^\infty dt\, e^{-\lambda t} T(t)$":

As a general strategy, we can always use the fact the A is closed and the definitions in Ex 1. are equivalent. Thus it suffices to find it, if it exists, on a dense subset. Again $\mathcal{S}$ is dense in $\mathcal{X}$ in both cases. Let $\lambda \in \mathbb{C}$. We try defining an operator on $\mathcal{S}$ by

$$\mathcal{S} \ni \varphi \longmapsto \int_0^\infty dt\, e^{-\lambda t} T(t) \varphi(\theta)$$

$$= \int_0^{-\theta} dt\, e^{-\lambda t} \varphi(\theta + t) , \qquad (\theta \in [-r,0)).$$

Clearly this defines a bounded operator in either $\mathcal{X}$-norm. Thus it can be uniquely extended to $\mathcal{X}$. Basically this means we can define $(\lambda - A)^{-1}$ by the Laplace T. $\int_0^\infty dt\, e^{-\lambda t} T(t)$ for all λ. This of course means $\lambda \in \rho(A)$ and $\rho(A) = \mathbb{C}$ since λ was arbit.

Remark: Both A and $(\lambda - A)^{-1}$ can be found by direct calc. using results from analysis. This, however, can nicely be avoided by using general semigroup results!

Ex 3 : First note the representation

$$T(t)f(x) = K_t * f(x), \qquad (x \in \mathbb{R}, t>0, f\in \mathcal{X}),$$

where $K_t(y) = e^{-y^2/2t}/\sqrt{2\pi t}$. The following properties are considered known:

$$\int_{\mathbb{R}} K_t(y)\,dy = 1, \ \forall t>0 \quad ; \quad \widehat{K_t}(p) = \frac{e^{-\frac{tp^2}{2}}}{\sqrt{2\pi}}.$$

To get started, the semigroup property $T(t+s)=T(t)T(s)$ follows, if we can show $\underline{K_{t+s} \overset{?}{=} K_t * K_s}$, $\forall t,s>0$.

By taking Fourier-T. the identity becomes

$$\frac{e^{-(t+s)p^2/2}}{\sqrt{2\pi}} = \widehat{K_{t+s}}(p) \overset{?}{=} \sqrt{2\pi}\,\widehat{K_t}(p)\,\widehat{K_s}(p)$$

$$= \sqrt{2\pi}\,\frac{e^{-tp^2/2}}{\sqrt{2\pi}} \cdot \frac{e^{-sp^2/2}}{\sqrt{2\pi}}, \quad (p\in\mathbb{R}),$$

which, it appears, is indeed true (so we can remove the "?").

<u>Strong continuity</u>:

As before, by equicontinuity of the family $\{T(t) : t\in[0,1]\}$, it is sufficient to show the strong continuity on a dense subset.

Schwartz class $\mathcal{S}(\mathbb{R})$ is dense in both $L^p(\mathbb{R})$ and $BUC(\mathbb{R})$. For any $\varphi\in\mathcal{S}$ we can safely use DCT to see

$$\int_{-\infty}^{\infty} dy \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} f(x-y) \overset{y=\sqrt{t}u}{=} \int_{-\infty}^{\infty} du \frac{e^{-u^2/2}}{\sqrt{2\pi}} f(x-\sqrt{t}u) \xrightarrow[t\to 0]{(DCT)} \int_{-\infty}^{\infty} du \frac{e^{-u^2/2}}{\sqrt{2\pi}} f(x) = f(x),$$

<u>uniformly</u> on x, by the unif. cont. of φ.

... Ex 3: That does it for $\mathcal{X} = BUC(\mathbb{R})$-case. For L^p one can use, for example, Jensen's ineq.:

$$\|K_t * \varphi - \varphi\|_p^p = \int dx \underbrace{\left| \int du \frac{e^{-u^2/2}}{\sqrt{2\pi}} \left(\varphi(x-\sqrt{t}u) - \varphi(x)\right) \right|^p}_{\leq \int du \frac{e^{-u^2/2}}{\sqrt{2\pi}} |\varphi(x-\sqrt{t}u) - \varphi(x)|^p}$$

$$(\text{Fubini}) \leq \int du \frac{e^{-u^2/2}}{\sqrt{2\pi}} \| \tau_{\sqrt{t}u} \varphi - \varphi \|_p^p$$

translation operator

Now $s \mapsto \tau_s \varphi$ is continuous in L^p norm ($p<\infty$), so that as $t \downarrow 0$ above, the last expression goes to 0 by DCT. And so $T(t)$ is strongly cont. in L^p.

Generator A:

First a bit of manipulation. Let $f \in \mathcal{X}$. Then

$$\frac{T(t)f(x) - f(x)}{t} = \int_{-\infty}^{\infty} \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} \left(\frac{f(x-y) - f(x)}{t}\right) dy$$

$$= \int_0^{\infty} \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} \left(\frac{f(x-y) + f(x+y) - 2f(x)}{t}\right) dy$$

$$(y = \sqrt{t}u) \quad = \int_0^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \underbrace{\left(\frac{f(x-\sqrt{t}u) + f(x+\sqrt{t}u) - 2f(x)}{t}\right)}_{\text{This looks like a "difference quotient" for the second derivative!}} du$$

Ex 3 That observation suggests we take $f \in \mathcal{X}$ such that $\exists f'' \in \mathcal{X}$. Then we have, in general, the identity

$$f(x-y) + f(x+y) - 2f(x) = \int_{-y}^{y} (y - |s|)\, f''(x+s)\, ds \;, \; (x, y \in \mathbb{R}).$$

Applying this to our problem gives

$$\frac{T(t)f - f}{t}(x) = \int_0^\infty du \frac{e^{-u^2/2}}{\sqrt{2\pi}} \underbrace{\frac{1}{t} \int_{-\sqrt{t}u}^{\sqrt{t}u} (\sqrt{t}u - |s|)\, f''(x+s)\, ds}$$

$$\overset{(s = uv)}{=} \frac{u^2}{t} \int_{-\sqrt{t}}^{\sqrt{t}} (\sqrt{t} - |v|)\, f''(x + vu)\, dv \;.$$

Now, using also $\int_0^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}} u^2 = 1$, we estimate

$$\left| \frac{T(t)f - f}{t}(x) - f''(x) \right| \le \int du \frac{e^{-u^2/2}}{\sqrt{2\pi}} u^2 \cdot \underbrace{\frac{1}{t} \int_{-\sqrt{t}}^{\sqrt{t}} (\sqrt{t} - |v|)\, |f''(x+uv) - f''(x)|\, dv}_{= \circledast} \;.$$

If $f'' \in BUC(\mathbb{R})$ then the term $\circledast$ goes to zero uniformly in x, implying $f'' = A_{BUC} f$.

If, on the other hand, $f'' \in L^p$, then we have

$$\left\| \frac{T(t)f - f}{t} - f'' \right\|_p^p \le \int dx \left| \int_0^\infty du \frac{e^{-u^2/2}}{\sqrt{2\pi}} \frac{1}{t} \int_{-\sqrt{t}}^{\sqrt{t}} (\sqrt{t} - |v|)(f''(x+uv) - f''(x))\, dv \right|^p$$

$$\text{(Double Jensen)} \le \int dx \int_0^\infty du \frac{e^{-u^2/2}}{\sqrt{2\pi}} \frac{1}{t} \int_{-\sqrt{t}}^{\sqrt{t}} (\sqrt{t} - |v|)\, |f''(x+uv) - f''(x)|^p\, dv$$

$$\text{(Double Fubini)} = \int_0^\infty du \frac{e^{-u^2/2}}{\sqrt{2\pi}} \frac{1}{t} \int_{-\sqrt{t}}^{\sqrt{t}} (\sqrt{t} - |v|)\, \| \tau_{uv} f'' - f'' \|_p^p$$

... Ex 3} By continuity of $t \mapsto T_t f''$ we see that the limit, as $t \to 0$, exists and is 0. Thus $f'' = A_{L^p} f$.

So far we have

1. $\mathcal{D}(A_{BUC}) \supseteq \{ f \in BUC : \exists f'' \in BUC \} = \mathcal{D}_1(\frac{d^2}{dx^2})$
2. $\mathcal{D}(A_{L^p}) \supseteq \{ f \in L^p : \exists f'' \in L^p \} = \mathcal{D}_2(\frac{d^2}{dx^2})$

We could go and check the other inclusion directly. However, recall that A is the generator of a contraction semigroup, so that $\rho(A)$ contains at least $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$. If we now can find just one $\mu > 0$ such that $\exists (\mu - \frac{d^2}{dx^2})^{-1} \in \mathcal{B}(\mathcal{X})$, it then follows A cannot be a proper extension of $\frac{d^2}{dx^2}$, for then $\mu - A \supsetneq \mu - d^2/dx^2 \Rightarrow (\mu - A)^{-1} \supsetneq (\mu - d^2/dx^2)^{-1}$ BUT $(\mu - d^2/dx^2)^{-1}$ ALREADY HAS FULL DOMAIN.

Thus we look for $\mu > 0$ s.t. $\mu \in \rho(\frac{d^2}{dx^2})$. First note that d^2/dx^2 is a closed operator (try it if you don't know it; it's easy to prove). Therefore it suffices to check conditions of Def. 1 in Ex 1.

(i) Injectivity: If $f, g \in \mathcal{D}(d^2/dx^2)$, $\mu \in \mathbb{C} \setminus (-\infty, 0]$ then

$(\mu - d^2/dx^2) f = (\mu - d^2/dx^2) g$

$\Leftrightarrow (\mu + p^2)\hat{f} = (\mu + p^2)\hat{g}$, as Schwartz distributions $\hat{f}, \hat{g} \in \mathcal{S}'$

$\Leftrightarrow \hat{f}((\mu + p^2)\varphi) = \hat{g}((\mu + p^2)\varphi) \quad \forall \varphi \in \mathcal{S}$

$\Leftrightarrow \hat{f} = \hat{g}$ since $\mu \notin (-\infty, 0]$

$\Leftrightarrow f = g$ Done.

... **Ex 3:** (ii) Condition $\overline{R(\mu - d^2/dx^2)} = \mathcal{H}$:

Let $\varphi \in \mathcal{S}$; then for $\mu \in \mathbb{C} \setminus (-\infty, 0]$

$$(\mu - d^2/dx^2)\psi = \varphi \iff (\mu + p^2)\hat{\psi} = \hat{\varphi}$$

$$\iff \hat{\psi}(p) = \frac{\hat{\varphi}(p)}{\mu + p^2}$$

$$\iff \psi(x) = \sqrt{2\pi}\left(\frac{1}{\mu + p^2}\right)^{\vee} * \varphi(x).$$

The expression for ψ shows ψ belongs to $\mathcal{H}$ at least, thus ensuring that $R(\mu - d^2/dx^2) \supseteq \mathcal{S}$ so we're done.

(iii) Boundedness of $(\mu - d^2/dx^2)^{-1} : R \longrightarrow D$:

It's sufficient to show boundedness on a dense subset $\mathcal{S}$.

This we do by completing the expression above.

One checks that for $\mu \in \mathbb{C} \setminus (-\infty, 0]$ we have

$$\left(\frac{1}{\mu + p^2}\right)^{\vee}(x) = \frac{1}{\mu}\left(\frac{1}{1 + (p/\sqrt{\mu})^2}\right)^{\vee}(x) = \sqrt{\frac{\pi}{2\mu}}\, e^{-\sqrt{\mu}|x|},$$

so we get as resolvent

$$(\mu - d^2/dx^2)^{-1}\varphi(x) = \psi(x) = \boxed{\frac{\pi}{\sqrt{\mu}}\, e^{-\sqrt{\mu}|y|} * \varphi(x)}, \quad (x \in \mathbb{R}),$$

which clearly defines a bounded operator.

This, then completes the proof that d^2/dx^2 is A and hence gives a representation for $(\lambda - A)^{-1}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ too.

... Ex 3: How about the values $\lambda \in (-\infty, 0]$?

Answer: Then $\lambda \notin \rho(A)$.

Proof: I'm afraid I'll resort to distributions again.

If $\lambda < 0$ Pick g such that $\hat{g}$ is e.g. some Gaussian

$\hat{g}(p) = \frac{e^{-(p-\sqrt{|\lambda|})^2/2}}{\sqrt{2\pi}}$, so that, and this is the point,

$\hat{g}(p) \geq \delta > 0$ "around" $\sqrt{|\lambda|}$. Then g surely belongs even to $\mathcal{S}$,

and yet if there were $f \in \mathcal{X}$ such that

$(\lambda - A) f = g$, then $(\lambda + p^2)\hat{f} = \hat{g}$, and

$\hat{f}(p) = \frac{\hat{g}(p)}{\lambda + p^2}$.

But such function $\hat{f}$ is not locally integrable and thus

$\hat{f} \notin \mathcal{S}'$ what it certainly did if $f \in \mathcal{X} \subset \mathcal{S}'$.

Hence no such $f \in \mathcal{X}$ exists and

$R(\lambda - A) \neq \mathcal{X}$ for $\lambda < 0$. $\square$

Ex 4: Let $x \in D(A^2)$. Then one can prove, just as in calculus, the "Taylor expansion"

$$T(t)x = x + tAx + \int_0^t ds\,(t-s)\,T(s)A^2 x \quad , (t \geq 0).$$

Now, using $\|T(t)\| \leq M$ we can estimate

$$\begin{aligned} M\|x\| &\geq \|T(t)x\| \\ &\geq -\|x\| + t\|Ax\| - \left\| \int_0^t ds\,(t-s)\,T(s)A^2 x \right\| \\ &\geq -\|x\| + t\|Ax\| - M\|A^2 x\| \underbrace{\int_0^t ds\,(t-s)}_{t^2/2} \end{aligned}$$

So that

$$\frac{M\|A^2x\|}{2} t^2 - \|Ax\|\,t + (M+1)\|x\| \geq 0$$

for all $t \geq 0$, and in fact all $\underline{t \in \mathbb{R}}$. Therefore the discriminant of the polynomial in t must be ≤ 0:

$$\|Ax\|^2 - 4 \cdot \frac{M\|A^2x\|}{2} \cdot (M+1)\|x\| \leq 0 .$$

This gives us

$$\|Ax\|^2 \leq 2 \cdot \underbrace{(M+1)}_{\leq 2 \cdot M} M \|A^2x\|\,\|x\| .$$

Ex 5: The left translation semigroup $\tau_t f(x) := f(x-t)$, $f \in BUC(\mathbb{R})$ is clearly strongly cont. by uniform cont. of the elements.

We show first that its generator A is $-\frac{d}{dx}$ with domain $D(A) = \{f \in BUC : f' \in BUC\}$:

Direction $D(A) \subseteq$ -"- is plain.

Let us then consider $f \in BUC$ such that $f' \in BUC$.

Then easily

$$\left\| \frac{f(x-t) - f(x)}{t} - f'(x) \right\|_\infty \leq \frac{1}{t} \int_{-t}^{0} \| f'(x+s) - f'(x) \|_\infty \, ds \xrightarrow{t \downarrow 0} 0,$$

since $f' \in BUC$. Thus $f \in D(A)$. Done.

Now we can apply Ex 4 to τ_t with $M = 1$.

In this case $f \in D(A^2)$ implies $Af = -f'$ and $A^2 f = f''$ so we have

$$\| f' \|_\infty^2 \leq 4 \, \| f'' \|_\infty \, \| f \|_\infty .$$

Btw. This makes sense, doesn't it?