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Perturbation Theory for Dual Semigroups

I. The Sun-Reflexive Case

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1. Introduction and an Outline of the Main Results

It is well-known that if A_0 is the infinitesimal generator of a C_0-semigroup of bounded linear operators $T_0(t)$ on a Banach space X, then given any bounded linear operator B on X, A_0+B generates a C_0-semigroup $T(t)$ on X. A proof of this fact exploits the variation-of-constants formula

$$T(t)x=T_0(t)x+\int_0^t T_0(t-\tau)BT(\tau)x\,d\tau \ , \tag{1.1}$$

for every $x\in X$ and $t\geqq 0$, to construct the "perturbed" semigroup $T(t)$.

In many concrete applications, e.g. delay equations and first-order partial functional-differential equations, we don't have quite this situation, yet the variation-of-constants formula plays a crucial role (although in a bit formal, as opposed to functional analytic, sense). In such cases B maps out of the space X into some "bigger" space Y, but convolution brings us back to X (see, for instance, Hale [8]).

The purpose of this paper is to introduce a systematic procedure to construct such a space Y and to give a precise, functional analytic, perturbation theory which serves as a unifying and simplifying framework for various special cases which arise in applications.

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Let $\langle , \rangle$ denote the pairing between X and its dual X^*. Then (1.1) is equivalent to:

$$\langle T(t)x, x^*\rangle = \langle T_0(t)x, x^*\rangle + \int_0^t \langle T_0(t-\tau)BT(\tau)x, x^*\rangle d\tau \ , \tag{1.2}$$

for every $x \in X$, $x^* \in X^*$ and $t \geqq 0$. When writing (1.2) one considers an element of X as a bounded linear functional on X^*, i.e. we embed X in X^{**} via the canonical isometry $i: X \to X^{**}$ defined by

$$i(x)(x^*) = \langle x, x^*\rangle \ , \quad \text{for every} \quad x \in X \quad \text{and} \quad x^* \in X^* \ .$$

In (1.2) one does not have to consider *all* $x^* \in X^*$. In fact (1.1) is still equivalent to

$$\langle T(t)x, z\rangle = \langle T_0(t)x, z\rangle + \int_0^t \langle T_0(t-\tau)BT(\tau)x, z\rangle d\tau \tag{1.3}$$

for every $x \in X$, $z \in Z$ and $t \geqq 0$, where Z is a weak * dense subspace of X^*. As we shall demonstrate, a natural choice for Z is the maximal invariant subspace of X^* on which $T_0^*(t)$ is strongly continuous. Here $T_0^*(t)$ is $T_0(t)^*$. We shall denote this subspace by $X^{\odot}$. It is known [2, 10, 17] that $X^{\odot}$ is closed in the norm topology of X^* and weak * dense in X^*.

Thus we consider an element of X as a bounded linear functional on $X^{\odot}$. Indeed, we can embed X into $X^{\odot *}$ via the continuous injection $j: X \to X^{\odot *}$ defined by

$$j(x)(x^{\odot}) = \langle x, x^{\odot}\rangle \quad \text{for every} \quad x \in X \quad \text{and} \quad x^{\odot} \in X^{\odot} \ .$$

It has been proved by Hille and Phillips [10] that

$$\|x\|' := \sup\{|\langle x, x^{\odot}\rangle| : x^{\odot} \in X^{\odot} \quad \text{and} \quad \|x^{\odot}\| \leqq 1\}$$

defines an equivalent norm on X (one even has $\|x\|' = \|x\|$ whenever $T_0(t)$ is a contraction semigroup). Therefore $j(X)$, the range of j in $X^{\odot *}$, is a closed subspace of $X^{\odot *}$. It is not difficult to verify that $j = p \circ i$, where $p: X^{**} \to X^{\odot *}$ is defined by

$$p(x^{**})(x^{\odot}) = \langle x^{\odot}, x^{**}\rangle \quad \text{for every} \quad x^{\odot} \in X^{\odot} \quad \text{and} \quad x^{**} \in X^{**} \ .$$

In other words, j is the original embedding i of X into X^{**}, but when one "ignores" the values of $i(x)$ outside of $X^{\odot}$.

As an instructive example, let X be $L^1(S^1)$ and $T_0(t)$ the group of translations. Then X^* can be identified with $L^\infty(S^1)$ and $X^{\odot}$ with $C(S^1)$. So via the embedding j, $L^1(S^1)$ will be considered as a closed subspace of $C(S^1)^*$ instead of $L^\infty(S^1)^*$.

If we denote by $T_0^{\odot}(t)$ the restriction of $T_0^*(t)$ to $X^{\odot}$ and by $T_0^{\odot *}(t)$ the adjoint of $T_0^{\odot}(t)$, then $T_0^{\odot *}(t)$ is a semigroup on $X^{\odot *}$ which is not necessarily strongly continuous. We shall denote by $X^{\odot\odot}$ the maximal invariant subspace of $X^{\odot *}$ on which $T_0^{\odot *}(t)$ is strongly continuous. In this paper we shall make the assumption

$$j(X) = X^{\odot\odot} \tag{1.4}$$

which is fulfilled in our example above (see [2, Sect. 1.4.2]). We call the case in which (1.4) holds the $\odot$-reflexive case (pronunciation: the sun-reflexive case; Amann [1] calls it the A_0-reflexive case, to emphasize that it is a property of the combination (X, A_0)). We shall recall below a useful compactness criterion for the resolvent of A_0

(due to Hille and Phillips [10]) which guarantees that (X, A_0) is $\odot$-reflexive. Whenever we have $\odot$-reflexivity we can (and will) identify X and $X^{\odot\odot}$ by means of j and then the situation can be conveniently summarized by the diagram

$$\begin{array}{ccc} X & \longrightarrow & X^* \\ \uparrow & & \downarrow \\ X^{\odot *} & \longleftarrow & X^{\odot} \end{array}$$

where a horizontal arrow indicates transition to the dual space (taking adjoints) and a vertical arrow taking restrictions to the maximal subspace of strong continuity. On each of the spaces we have a semigroup and a generator indexed by the same symbols as the space and related to each other as indicated by the arrows.

Rewriting (1.3), with $Z = X^{\odot}$, as

$$\langle T(t)x, x^{\odot}\rangle = \langle T_0(t)x, x^{\odot}\rangle + \int_0^t \langle BT(\tau)x, T_0^{\odot}(t-\tau)x^{\odot}\rangle d\tau \ , \qquad (1.5)$$

for every $x^{\odot} \in X^{\odot}$ and $x \in X$"="$X^{\odot\odot}$ and $t \geqq 0$, we find that the equation still makes sense if B maps X continuously into $X^{\odot *}$. The main point of this paper is to prove that in this case (1.5) uniquely defines a C_0 semigroup $T(t)$ on X and to give a characterization of its generator. So here the "bigger" space Y is chosen to be $X^{\odot *}$.

Note that if X is reflexive, i.e. $i(X) = X^{**}$, then it is known [2, 10, 17] that $X^{\odot} = X^*$ and consequently $X^{\odot *} = X^{**} = X$. So in this case the results of this paper reduce to the well-known results concerning bounded perturbations of the generator. Therefore we concentrate on the case in which X is not reflexive.

We shall prove that $X^{\odot}$ is also the maximal invariant subspace on which $T^*(t)$ is strongly continuous and that $X = X^{\odot\odot}$ is the maximal invariant subspace on which $T^{\odot *}(t)$ is strongly continuous (and that $T^{\odot\odot}(t) = T(t)$). Since both $T_0^{\odot *}(t)$ and $T^{\odot *}(t)$ are adjoints of strongly continuous semigroups one can define their weak * generators $A_0^{\odot *}$ and $A^{\odot *}$. We shall prove that $D(A^{\odot *}) = D(A_0^{\odot *})$. This implies, in particular, that the domain of the weak * generator of the perturbed semigroup is *independent* of the perturbation B. However, the generator A of $T(t)$ turns out to be the part of $A^{\odot *}$ in X and hence $D(A)$ will, in general, depend on B. This may even go so far that *all* information about B is contained in $D(A)$ (i.e. the action of A_0 is the same as the action of A but their domains are different). So by considering a duality framework with four spaces we accomplish the realization of two desirable properties: domain independence on the big spaces and strong continuity on the small spaces.

In this paper we restrict ourselves to linear problems. It should be clear, however, that the domain independence on the big space is the key point for a theory concerning *semilinear* problems. A paper on this subject, dealing with such items as linearized stability, the center manifold and Hopf bifurcation, is in preparation. In fact the original motivation for this work comes from attempts to treat nonlinear structured population problems (see, for instance, Gyllenberg [7], Heijmans [9], Metz and Diekmann [12] and Thieme [16]) as semilinear evolution equations.

In the applications our approach is in the spirit of older work of Kappel and Schappacher [11] on delay equations and more recent work of Desch and

Schappacher [4] on, among other things, age dependent population dynamics: take for $T_0(t)$ the simplest *prototype* semigroup in the category one is interested in. For example, the semigroup corresponding to $\dot{x}=0$ considered as a delay equation and, more generally, the semigroup obtained by neglecting (i.e. putting equal to zero) all *non-local* terms (see Suhadolc and Vidav [15] for an example from transport theory; it seems likely that our framework covers such examples as well). Greiner [6] uses a different approach. He obtains a new generator by perturbing the domain of the old one. Actually, our results unify the case in which generators act differently on identical domains with cases in which they act identically on different domains.

The theory of this paper will be illustrated by means of the equation modelling age-dependent population growth. In a separate paper Diekmann [5] will deal with delay equations.

We conclude this Introduction with some remarks about notation. Elements of $X, X^*, X^\odot$ etc. are denoted by $x, x^*, x^\odot$ etc. We use $\langle x, x^*\rangle$ and $\langle x^*, x\rangle$ interchangeably to denote $x^*(x)$, i.e. the value of x^* at x, whenever $x\in X$ and $x^*\in X^*$.

2. Dual Semigroups

Let X be a (non-reflexive) Banach space and let $T(t)$ be a strongly continuous semigroup of bounded linear operators on X with infinitesimal generator A. Let $T^*(t)$ denote the semigroup of adjoint operators acting on the dual space X^* and let A^* denote the adjoint of A. The following theorems summarize some well known results. Proofs may be found in Butzer and Berens [2], Yosida [17], Hille and Phillips [10]. Also see Amann [1].

Theorem 2.1. (i) *For any $x^*\in X^*$ the map $t\to T^*(t)x^*$ from $\mathbb{R}^+$ into X^* equipped with the weak * topology is continuous.*

(ii) *A^* is the weak * generator of $T^*(t)$, i.e. x^* belongs to $D(A^*)$ iff $\frac{1}{t}(T^*(t)x^*-x^*)$ converges in the weak * topology as $t\downarrow 0$, and whenever there is convergence the limit equals A^*x^*.*

(iii) *If x^* belongs to $D(A^*)$ so does $T^*(t)x^*$ for any $t\geqq 0$ and $A^*T^*(t)x^* = T^*(t)A^*x^*$.*

Theorem 2.1 (iii) expresses that $u^*(t)=T^*(t)x^*$ is a solution of the differential equation

$$\frac{d}{dt}u^*(t)=A^*u^*(t)\ ,\qquad u^*(0)=x^*\ ,$$

whenever $x^*\in D(A^*)$, though differentiation has to be understood in the weak * sense.

Unless X^* is reflexive, the semigroup $T^*(t)$ need not be strongly continuous if we equip X^* with its norm topology. Because of Theorem 2.1 (i) we call $T^*(t)$ a weak * continuous semigroup. The following definition makes sense.

Definition 2.2. $X^\odot=\{x^*\in X^*:\lim\limits_{t\downarrow 0}\|T^*(t)x^*-x^*\|=0\}$.

Clearly the subspace $X^{\odot}$ is invariant under $T^*(t)$ and it is easily proved that $X^{\odot}$ is norm-closed. Let $T^{\odot}(t)$ denote the restriction of $T^*(t)$ to $X^{\odot}$. Then $T^{\odot}(t)$ is strongly continuous. Let $A^{\odot}$ denote its generator.

Theorem 2.3 (Phillips). (i) $X^{\odot}=\overline{D(A^*)}$.

(ii) *$A^{\odot}$ is the part of A^* in $X^{\odot}$, i.e. the largest restriction of A^* with both domain and range in $X^{\odot}$.*

(iii) *$D(A^{\odot})$ is weak * dense in X^*.*

Next we present some less known results which constitute the essential part of Chapter XIV of Hille and Phillips [10]. By definition

$$\|x^{\odot}\|=\sup\{|\langle x,x^{\odot}\rangle|:x\in X,\ \|x\|\leqq 1\}\ ,$$

for $x^{\odot}\in X^{\odot}$. In order to enhance the symmetry we introduce

$$\|x\|'=\sup\{|\langle x,x^{\odot}\rangle|:x^{\odot}\in X^{\odot},\ \|x^{\odot}\|\leqq 1\}\ ,$$

for $x\in X$.

Lemma 2.4. (i) $\|x\|'\leqq\|x\|\leqq M\|x\|'$ *where*

$$M=\lim_{\lambda\to\infty}\inf\ \|\lambda(\lambda I-A)^{-1}\|<\infty\ .$$

In other words, $\|\cdot\|'$ is a norm equivalent with the original norm and when $T(t)$ is a contraction semigroup the two norms are actually the same.

(ii) *If we equip X with the prime norm, the norm on $X^{\odot}$ remains unchanged, i.e.*

$$\|x^{\odot}\|=\sup\{|\langle x,x^{\odot}\rangle|:x\in X,\ \|x\|'\leqq 1\}\ .$$

On $X^{\odot *}$ we have, by duality, a weak * continuous semigroup $T^{\odot *}(t)$ with weak * generator $A^{\odot *}$. Every element of X defines a continuous linear functional on X^*, so a fortiori on $X^{\odot}$, and therefore can be considered as an element of $X^{\odot *}$. If $\langle x_1-x_2,x^{\odot}\rangle=0$ for all $x^{\odot}\in X^{\odot}$ then necessarily $x_1=x_2$, since $X^{\odot}$ is weak * dense in X^* and X^* separates the points of X. So if we equip X with the prime norm there exists an isometric isomorphism of X onto a closed subspace of $X^{\odot *}$, i.e. we can embed X into $X^{\odot *}$ by means of the natural mapping. We shall, from now on, identify X with its embedding into $X^{\odot *}$.

Continuing our game of taking restrictions we introduce

$$X^{\odot\odot}=\{x^{\odot *}\in X^{\odot *}:\lim_{t\downarrow 0}\|T^{\odot *}(t)x^{\odot *}-x^{\odot *}\|=0\}\ .$$

There is no need for a new norm on $X^{\odot}$, as Hille and Phillips prove:

Lemma 2.5. *The prime norm on $X^{\odot}$ is the same as the original norm.*

Since $T(t)$ is strongly continuous we clearly have that $X\subset X^{\odot\odot}$.

Definition 2.6. X is called $\odot$-reflexive with respect to A iff $X=X^{\odot\odot}$.

Theorem 2.7. *X is $\odot$-reflexive with respect to A iff $(\lambda I-A)^{-1}$ is $\sigma(X,X^{\odot})$-weakly compact for $\lambda\in\varrho(A)$.*

Theorem 2.8. *X is $\odot$-reflexive with respect to A iff $X^{\odot}$ is $\odot$-reflexive with respect to $A^{\odot}$.*

3. The Variation-of-Constants Formula

In this section, we shall prove that (1.5) can be rewritten as an equation in X (identified with $j(X)=X^{\odot\odot}$):

$$T(t)x=T_0(t)x+\int_0^t T_0^{\odot *}(t-\tau)BT(\tau)x\,d\tau\ , \qquad t\geqq 0, x\in X\ . \tag{3.1}$$

Therefore we have to give a precise meaning to the integral term. Since we are looking for a strongly continuous semigroup $T(t)$ on X, the function $\tau\to T(\tau)x$ will be a continuous function from $[0,t]$ in X, which we denote by u. Bu then is a continuous function from $[0,t]$ in $X^{\odot *}$. So we are led to define an integral of the form

$$\int_0^t T^{\odot *}(t-\tau)f(\tau)\,d\tau\ , \qquad t\geqq 0 \tag{3.2}$$

where f is continuous from $[0,t]$ in $X^{\odot *}$. We shall even consider the case where $f\colon[0,t]\to X^{\odot *}$ is only weak * continuous.

First we describe how to integrate a weak * continuous function. Let Z be a Banach space and Z^* its dual. Let $z^*(t)$ be a weak * continuous function from an interval $[a,b]$ into Z^*. Then the integral $\int_a^b \langle z, z^*(t)\rangle dt$ makes sense for any $z\in Z$, and satisfies

$$\left|\int_a^b \langle z,z^*(t)\rangle dt\right| \leqq (b-a)\|z\|\cdot \sup_{t\in[a,b]}\|z^*(t)\|\ .$$

Recall that, by the uniform boundedness theorem, any weak * continuous function is norm bounded. So we get that

$$z\to\int_a^b \langle z,z^*(t)\rangle dt$$

defines a continuous linear functional on Z, hence an element of Z^*, which we denote by $\int_a^b z^*(t)dt$. Note that by definition

$$\left\langle z, \int_a^b z^*(t)dt\right\rangle = \int_a^b \langle z,z^*(t)\rangle dt\ , \qquad \text{for all} \quad z\in Z\ .$$

If L is a bounded linear operator on Z, then $t\to L^*z^*(t)$ is weak * continuous as well, and it is easy to show that

$$\int_a^b L^*z^*(t)dt = L^*\int_a^b z^*(t)dt\ .$$

In general we may not replace L^* by some bounded linear operator on Z^* which is not a dual operator.

Now, let us return to the integral in (3.2) with f weak * continuous. For fixed $t\geqq 0$, the function $\tau\to T^{\odot *}(t-\tau)f(\tau)$ is weak * continuous on $[0,t]$, and the integral can be defined as above. Assume that $M \geqq 1$ and $\omega \in \mathbb{R}$ are chosen such that

$$\|T(t)\| \leqq Me^{\omega t} , \quad t \geqq 0 . \tag{3.3}$$

Then we have the following estimate.

Lemma 3.1. $\left\|\int_0^t T^{\odot *}(t-\tau)f(\tau)d\tau\right\| \leqq M \cdot \frac{e^{\omega t}-1}{\omega} \cdot \sup_{0 \leqq \tau \leqq t} \|f(\tau)\|$.

Here $\frac{e^{\omega t}-1}{\omega}$ is to be interpreted as t if $\omega = 0$.

We can prove a lot more if f is norm-continuous.

Theorem 3.2. *Let f: $[0, \infty) \to X^{\odot *}$ be norm-continuous, then $t \to \int_0^t T^{\odot *}(t-\tau)f(\tau)d\tau$ is a norm-continuous $X^{\odot\odot}$-valued function.*

Proof. For $t \geqq 0$ we define $F(t) = \int_0^t T^{\odot *}(t-\tau)f(\tau)d\tau$.

(i) We first show that $F(t) \in X^{\odot\odot}$, for $t \geqq 0$. Let $h > 0$. Then

$$\begin{aligned} T^{\odot *}(h)F(t) - F(t) &= \int_0^t T^{\odot *}(t+h-\tau)f(\tau)d\tau - \int_0^t T^{\odot *}(t-\tau)f(\tau)d\tau \\ &= -\int_0^h T^{\odot *}(\tau)f(t-\tau)d\tau + \int_t^{t+h} T^{\odot *}(\tau)f(t+h-\tau)d\tau \\ &\quad + \int_h^t T^{\odot *}(\tau)\{f(t+h-\tau) - f(t-\tau)\}d\tau . \end{aligned}$$

The norms of the first two terms are less than

$$Kh \sup_{0 \leqq \tau \leqq t+h} \|f(\tau)\| ,$$

for K large enough. The norm of the last term is less than

$$K \sup_{0 \leqq \tau \leqq t} \|f(\tau+h) - f(\tau)\| .$$

Therefore $T^{\odot *}(h)F(t) \to F(t)$, as $h \downarrow 0$, and by definition $F(t) \in X^{\odot\odot}$.

(ii) Since

$$F(t+h) - F(t) = T^{\odot *}(h)F(t) - F(t) + \int_0^h T^{\odot *}(\tau)f(t+h-\tau)d\tau$$

and

$$F(t-h) - F(t) = \int_0^h T^{\odot *}(t-\tau)f(\tau)d\tau + \int_h^t T^{\odot *}(t-\tau)\{f(\tau-h) - f(\tau)\}d\tau$$

it follows that $X^{\odot\odot}$-valuedness implies norm-continuity. □

4. Perturbation Theory

Consider a strongly continuous semigroup $T_0(t)$ on X with generator A_0. We will refer to these as the *unperturbed* semigroup and generator. From now on we make the basic.

Assumption 4.1. X is $\odot$-reflexive with respect to A_0.

The *perturbation* is defined, on the level of the generator, by a bounded linear operator B from X into $X^{\odot *}$. The adjoint of B maps $X^{\odot **}$ into X^* but we will only consider its restriction to $X^\odot$ and write $B^*: X^\odot \to X^*$. The basic idea now is to construct a semigroup $T(t)$ on X by solving the variation-of-constants equation

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(t-\tau)BT(\tau)x\,d\tau \tag{4.1}$$

by successive approximations. We call $T(t)$ the *perturbed* semigroup. After the preparatory work of Sect. 3 the proof that this method works is identical to the one for a truly bounded perturbation as given in, for instance,Pazy [13; Sect. 3.1] or Davies [3; Sect. 3.1].

Theorem 4.2. *Equation* (4.1) *uniquely defines a strongly continuous semigroup* $T(t)$ *on* X. *The successive approximations converge in the uniform operator topology, uniformly for* t *in compact sets.*

The determination of the generator A is rather simple if we make a detour via X^*, $X^\odot$ and $X^{\odot *}$. But first we make sure that $X^\odot$ is the space of strong continuity of $T^*(t)$ as well. Let us define

$$U(t) = \int_0^t T_0^{\odot *}(t-\tau)BT(\tau)\,d\tau \tag{4.2}$$

Lemma 3.1 implies that, as $t \downarrow 0$, $\|U(t)\| \to 0$ and therefore $\|U^*(t)\| \to 0$ as well. Hence we have

Lemma 4.3. *The function* $t \to T^*(t)x^*$ *is norm-continuous on* $\mathbb{R}_+$ *if and only if* $x^* \in X^\odot$.

So if $x^* \in D(A^*)$ then necessarily $x^* \in X^\odot$ and in order to characterize $D(A^*)$ we may restrict our attention to elements of $X^\odot$.

Lemma 4.4. *Let* $x^\odot \in X^\odot$, *then* $\frac{1}{t} U^*(t)x^\odot \to B^*x^\odot$ *as* $t \downarrow 0$, *relative to the weak * topology.*

Proof. $\frac{1}{t}\int_0^t \langle T(\tau)x, B^*T_0^\odot(t-\tau)x^\odot\rangle d\tau \to \langle x, B^*x^\odot\rangle$ as $t \downarrow 0$. □

Recalling Theorem 2.1 (ii) we infer that

Corollary 4.5. $D(A^*) = D(A_0^*)$ *and for* $x^\odot \in D(A^*)$ *we have* $A^*x^\odot = A_0^*x^\odot + B^*x^\odot$.

From Theorem 2.3 (ii) it then follows that

Corollary 4.6. $D(A^{\odot})=\{x^{\odot}\in D(A_0^*): A_0^*x^{\odot}+B^*x^{\odot}\in X^{\odot}\}$ *and* $A^{\odot}x^{\odot}=A_0^*x^{\odot}+B^*x^{\odot}$.

Since both $T^*(t)$ and $T_0^*(t)$ leave $X^{\odot}$ invariant the same must be true for $U^*(t)$. Let $U^{\odot}(t)$ denote the restriction. Clearly

$$U^{\odot}(t)=\int_0^t T^*(\tau)B^*T_0^{\odot}(t-\tau)d\tau \ , \tag{4.3}$$

where the integral is defined as before (i.e. as an X^*-valued function which, actually, takes values in the subspace $X^{\odot}$). Since $U^{\odot}(t)$ is the restriction of $U^*(t)$ necessarily $\|U^{\odot}(t)\|\to 0$ as $t\downarrow 0$ and likewise $\|U^{\odot *}(t)\|\to 0$. Therefore the mapping $t\to T^{\odot *}(t)x^{\odot *}$ is norm-continuous iff $t\to T_0^{\odot *}(t)x^{\odot *}$ is norm-continuous. This yields:

Theorem 4.7. *X is $\odot$-reflexive with respect to A.*

Exactly as above we deduce

Lemma 4.8. *As $t\downarrow 0$ then $\frac{1}{t}U^{\odot *}(t)x\to Bx$ in the weak * topology.*

Corollary 4.9. *$D(A^{\odot *})=D(A_0^{\odot *})$ and $A^{\odot *}=A_0^{\odot *}+B$.*

Corollary 4.10. *$D(A)=\{x\in D(A_0^{\odot *}): A_0^{\odot *}x+Bx\in X\}$, and $Ax=A_0^{\odot *}x+Bx$.*

Remarks. (i) We may as well start with $T_0^{\odot}(t)$ as the unperturbed semigroup and use

$$S^{\odot}(t)=T_0^{\odot}(t)+\int_0^t T_0^*(t-\tau)B^*S^{\odot}(\tau)d\tau \ . \tag{4.4}$$

Everything goes in exactly the same way and, in particular, we obtain the same generators. Hence $S^{\odot}(t)=T^{\odot}(t)$ and as an alternative to (4.2) we find

$$U(t)=\int_0^t T^{\odot *}(t-\tau)BT_0(\tau)d\tau \ , \tag{4.5}$$

with the role of perturbed and unperturbed semigroups interchanged.

(ii) Actually, assumption 4.1 is not needed for the construction of the solution $S^{\odot}(t)$ of (4.4)!

(iii) In the present presentation all proofs are based on an examination of the behaviour of the semigroups for $t\downarrow 0$. Alternative proofs would exploit the behaviour of the resolvents of the generators for $\lambda\to\infty$, and the "variation-of-constants formula" for the resolvents

$$R(\lambda,A)=R(\lambda,A_0)+R(\lambda,A_0^{\odot *})BR(\lambda,A) \ .$$

Note that for $\lambda\in\varrho(A_0)$, $R(\lambda,A_0^{\odot *})B$ is a bounded linear operator from X into X.

(iv) The Favard class of a C_0-semigroup on X is the set of $x\in X$ which yield Lipschitz continuous orbits under the semigroup. Known results (see Theorem 2.1.4 in [2]) imply that $D(A_0^{\odot *})$ is the Favard class of both $T_0(t)$ and $T(t)$. This also shows the connection between the Favard class and the generalized domain of the generator.

5. Perturbations with Finite Dimensional Range

In several applications of wide interest the operator B has finite dimensional range, so it seems appropriate to elaborate on this special case. Our presentation has points in common with some of the work by Desch and Schappacher [4].

Let there be given $r_1^{\odot *}, \ldots, r_n^{\odot *} \in X^{\odot *}$ and $r_1^*, \ldots, r_n^* \in X^*$ such that

$$Bx = \sum_{i=1}^{n} \langle r_i^*, x \rangle r_i^{\odot *} . \tag{5.1}$$

We define the entries $q_{ij}(t)$ of the matrix-valued function Q by

$$q_{ij}(t) = \left\langle r_i^*, \int_0^t T_0^{\odot *}(\tau) r_j^{\odot *} d\tau \right\rangle . \tag{5.2}$$

The estimate

$$|q_{ij}(t_1) - q_{ij}(t_2)| \leqq \frac{M}{\omega} (e^{\omega t_2} - e^{\omega t_1}) \|r_i^*\| \, \|r_j^{\odot *}\|$$

shows that Q is locally Lipschitz continuous and we conclude that Q has a representation of the form

$$Q(t) = \int_0^t K(\tau) d\tau , \tag{5.3}$$

where the entries k_{ij} of K belong to L_∞^{loc}.

Lemma 5.1. *For any integrable* $\mathbb{R}$*-valued function* η *the identity*

$$\left\langle r_i^*, \int_0^t T_0^{\odot *}(t-\tau) r_j^{\odot *} \eta(\tau) d\tau \right\rangle = \int_0^t k_{ij}(t-\tau) \eta(\tau) d\tau$$

holds.

Proof. Equality holds for $t=0$. As r_i^* does in general not commute with the integral, we integrate both sides of the above equation. Integrating the left-hand side we find

$$\begin{aligned} \int_0^t \left\langle r_i^*, \int_0^s T_0^{\odot *}(s-\tau) r_j^{\odot *} \eta(\tau) d\tau \right\rangle ds &= \left\langle r_i^*, \int_0^t \int_0^s T_0^{\odot *}(s-\tau) r_j^{\odot *} \eta(\tau) d\tau ds \right\rangle \\ &= \left\langle r_i^*, \int_0^t \int_0^{t-\tau} T_0^{\odot *}(\sigma) d\sigma r_j^{\odot *} \eta(\tau) d\tau \right\rangle \\ &= \int_0^t \left\langle r_i^*, \int_0^{t-\tau} T_0^{\odot *}(\sigma) d\sigma r_j^{\odot *} \right\rangle \eta(\tau) d\tau \\ &= \int_0^t q_{ij}(t-\tau) \eta(\tau) d\tau , \end{aligned}$$

and clearly integration of the right-hand side yields the same result. □

Now let, as before, $T(t)x$ be defined by the variation-of-constants equation

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(t-\tau) B T(\tau) x d\tau , \tag{5.4}$$

and define the n-vector $y(t)$ by

$$y_i(t) = \langle r_i^*, T(t)x \rangle \ . \tag{5.5}$$

Then (5.4) and Lemma 5.1 together imply that y satisfies the *renewal equation*

$$y = h + K * y \ , \tag{5.6}$$

where the n-vector valued forcing function h is given by

$$h_i(t) = \langle r_i^*, T_0(t)x \rangle \ , \tag{5.7}$$

and $K * y$ denotes the convolution product of K and y. Conversely, given any solution y of (5.6) with h of the specific form (5.7) we can recover $T(t)x$ by rewriting (5.4) in the form

$$T(t)x = T_0(t)x + \sum_{j=1}^{n} \int_0^t T_0^{\odot *}(t-\tau) r_j^{\odot *} y_j(\tau) d\tau \ . \tag{5.8}$$

We conclude that the "projected" renewal equation (5.6) contains all the information!

Remarks. (i) We have chosen the indirect definition of the kernel K via (5.3) since a direct definition seems impossible in general.

(ii) Since

$$B^* x^\odot = \sum_{i=1}^{n} \langle x^\odot, r_i^{\odot *} \rangle r_i^* \tag{5.9}$$

and

$$T^\odot(t) = T_0^\odot(t) + \int_0^t T_0^*(t-\tau) B^* T^\odot(\tau) d\tau \ , \tag{5.10}$$

we find that the n-vector valued function z defined by

$$z_i(t) = \langle T^\odot(t) x^\odot, r_i^{\odot *} \rangle \tag{5.11}$$

satisfies the "adjoint" renewal equation

$$z = g + K^T * z \tag{5.12}$$

where K^T denotes the transpose of the matrix K (if the entries are complex we have to take complex conjugates as well) and the forcing function g is defined by

$$g_i(t) = \langle T_0^\odot(t) x^\odot, r_i^{\odot *} \rangle \ . \tag{5.13}$$

6. Age-Dependent Population Dynamics

In order to illustrate the theory developed so far we apply it to a well-known example from structured population dynamics. We do not present any new results, but intend to demonstrate the usefulness of the new functional analytic framework. In order to remain in the $\odot$-reflexive domain we consider a population which is distributed over a *finite* age interval. In a follow-up we will show how the results extend, mutatis mutandae, to distributions over $\mathbb{R}_+$. Moreover, we plan to elaborate the basic ideas much further, and to demonstrate how the resulting approach opens the way to a rather general theory for both linear and nonlinear systems describing structured populations.

Before we start, let us recall some facts from measure theory. Let $\mathscr{B}$ denote the σ-algebra of all Borel sets in [0, 1], and let $M[0,1]$ be the space of all complex regular Borel measures on [0, 1]. Then $M[0,1]$ can be identified with the dual space of $C[0,1]$ by using the pairing $\langle\phi,\mu\rangle=\int\phi d\mu$. With $M_{AC}[0,1]$ we denote the subspace consisting of all absolutely continuous Borel measures on [0, 1]. For every $\mu\in M_{AC}[0,1]$ there exists a unique $h_\mu\in L^1[0,1]$ such that for every Borel set Ω in $\mathscr{B}$,

$$\mu(\Omega)=\int_\Omega h_\mu d\lambda \ ,$$

where λ represents the Lebesgue measure. h_μ is called the Radon-Nikodym derivative of μ with respect to the Lebesgue-measure. It is clear that $\mu\leftarrow\rightarrow h_\mu$ defines an isomorphism between $M_{AC}[0,1]$ and $L^1[0,1]$. For more details we refer to Rudin [14].

Let $\beta\in L^\infty[0,1]$. Consider the initial value problem

$$\frac{\partial m}{\partial t}(t,a)-\frac{\partial m}{\partial a}(t,a)=\beta(a)m(t,0) \ , \quad 0<a<1,\ t>0 \tag{6.1a}$$

$$m(t,1)=0 \ , \quad t\geqq 0 \tag{6.1b}$$

$$m(0,a)=\phi(a) \ , \quad 0\leqq a\leqq 1 \ . \tag{6.1c}$$

Here ϕ is a continuous function on [0, 1] with $\phi(1)=0$, and hence (6.1b) is satisfied for $t=0$. Let

$$X=C_0[0,1]=\{\phi\in C[0,1]:\phi(1)=0\} \ .$$

If we equip as usual X with the supremum norm then it becomes a Banach space. Readers who are familiar with age-dependent population models, might be surprized that we start with (6.1). However, the forthcoming analysis makes this understandable. We can rewrite (6.1), with (6.1a) replaced by

$$\frac{\partial m}{\partial t}(t,a)-\frac{\partial m}{\partial a}(t,a)=0 \ , \quad 0<a<1,\ t>0 \ ,$$

as the abstract Cauchy problem

$$\frac{dm}{dt}(t)=A_0m(t) \ , \quad m(0)=\phi \ , \tag{6.2}$$

where A_0 is the closed operator

$$A_0\phi=\phi' \ ,$$

for every ϕ in the domain given by

$$D(A_0)=\{\phi\in C^1[0,1]:\phi(1)=\phi'(1)=0\} \ .$$

It is easily seen that A_0 generates a strongly continuous semigroup $T_0(t)$ explicitly given by

$$(T_0(t)\phi)(a)=\begin{cases}\phi(a+t) & a+t\leqq 1 ,\\ 0 & a+t>1 .\end{cases}$$

On the dual space

$$X^*=\{\mu\in M[0,1]:\mu(\{1\})=0\} ,$$

the dual (weak * continuous) semigroup $T_0^*(t)$ is given by

$$(T_0^*(t)\mu)(\Omega)=\mu(\Omega_t) , \qquad \Omega\in\mathscr{B} ,$$

where the Borel set Ω_t is $\{\omega+t:\omega\in\Omega\}\cap[0,1]$. Its weak * generator A_0^* has domain

$$D(A_0^*)=\{\mu\in M_{AC}[0,1]:h_\mu(a)=v_\mu([0,a)), a\in[0,1], \text{ for some } v_\mu\in X^*\} ,$$

and is given by

$$A_0^*\mu=-v_\mu$$

for $\mu\in D(A_0^*)$. We recall that h_μ is the Radon-Nikodym derivative of μ with respect to the Lebesgue measure. Now obviously

$$X^\odot=\overline{D(A_0^*)}=M_{AC}[0,1] ,$$

and, as we noted above, we can identify $M_{AC}[0,1]$ with $L^1[0,1]$. So we take the representation

$$X^\odot=L^1[0,1] .$$

From the above expressions for A_0^* and $D(A_0^*)$ we derive that

$$D(A_0^\odot)=\{\psi\in AC[0,1]:\psi(0)=0\} ,$$

and for $\psi\in D(A_0^\odot)$,

$$A_0^\odot\psi=-\psi' .$$

The Cauchy problem

$$\frac{dn}{dt}(t)=A_0^\odot n(t) , \qquad n(0)=\psi ,$$

where $\psi\in X^\odot$, is an abstract representation of the initial value problem

$$\frac{\partial n}{\partial t}(t,a)+\frac{\partial n}{\partial a}(t,a)=0 , \qquad t>0,\ 0<a<1 \tag{6.3a}$$

$$n(t,0)=0 , \qquad t\geqq 0 \tag{6.3b}$$

$$n(0,a)=\psi(a) , \qquad 0\leqq a\leqq 1 . \tag{6.3c}$$

If we interpret a as age, and $n(t,a)$ as a density for individuals with age a at time t, then (6.3) is the system describing the evolution of a population with age-structure, where no births and no deaths occur.

Taking duals once more we get

$$X^{\odot *}=L^{\infty}[0,1] ,$$

$$(T_0^{\odot *}(t)\phi)(a)=\begin{cases}\phi(a+t) & a+t\leqq 1\\ 0 & a+t>1\end{cases}$$

$$D(A_0^{\odot *})=\{\phi\in \mathrm{Lip}\,[0,1]:\phi(1)=0\}$$

$$A_0^{\odot *}\phi=\phi' .$$

Here Lip [0, 1] consists of all functions which are Lipschitz continuous on [0, 1].

One easily sees that $X^{\odot\odot}=\overline{D(A_0^{\odot *})}=X=C_0[0,1]$, and therefore X is $\odot$-reflexive with respect to A_0. We note that $\odot$-reflexivity also follows from the compactness of the resolvent $R(\lambda, A_0)$. Let $B:X\to X^{\odot *}$ be the perturbation

$$(B\phi)(a)=\beta(a)\phi(0) .$$

If A is the operator as determined by Corollary 4.10, i.e.

$$D(A)=\{\phi\in \mathrm{Lip}\,[0,1]:\phi(1)=0 \text{ and } \phi'+\phi(0)\beta\in C_0[0,1]\}$$

$$A\phi=\phi'+\phi(0)\beta ,$$

then the abstract representation of (6.1) is,

$$\frac{dm}{dt}(t)=Am(t) , \quad m(0)=\phi ,$$

as it should be. It is easy to check that the abstract Cauchy problem

$$\frac{dn}{dt}(t)=A^{\odot}n(t) , \quad n(0)=\psi ,$$

represents the partial differential equation,

$$\frac{\partial n}{\partial t}(t,a)+\frac{\partial n}{\partial a}(t,a)=0 , \quad t>0,\ 0<a<1 , \tag{6.4a}$$

with boundary condition

$$n(t,0)=\int_0^1 \beta(a)n(t,a)da , \quad t\geqq 0 , \tag{6.4b}$$

and initial condition

$$n(0,a)=\psi(a) , \quad 0\leqq a\leqq 1 . \tag{6.4c}$$

This latter system governs the evolution of a population with age-structure, whose per capita birth rate is $\beta(a)$. The boundary condition expresses the fact that all newborns have age zero. Our abstract theory (Sect. 4) tells us that both A and $A^{\odot}$ generate strongly continuous semigroups $T(t)$ and $T^{\odot}(t)$ respectively.

Because of the symmetry we might as well have chosen the system (6.4) as our starting point. In that case the perturbation $C:X^{\odot}\to X^{*}$ looks as follows

$$(C\psi)(a)=\int_0^1 \beta(a)\psi(a)da\cdot\delta ,$$

where $\delta \in M[0,1]$ represents the (Dirac) measure concentrated at $a=0$. Here C is the restriction of $B^*: X^{\odot **} \to X^*$ to $X^{\odot}$. Because B and C have a one-dimensional range we can go one step further, and apply the results of Sect. 5. Let Q be the scalar valued function

$$Q(t)=\left\langle \beta, \int_0^t T_0^*(s)\delta ds \right\rangle .$$

Clearly

$$\left(\int_0^t T_0^*(s)\delta ds\right)(a)=\begin{cases}1, & \text{if } a<t ,\\ 0, & \text{otherwise} .\end{cases}$$

From this we obtain immediately that

$$Q(t)=\int_0^t \beta(a)da ,$$

and therefore

$$K(t)=\beta(t) .$$

For $y(t)=\langle \beta, n(t)\rangle =\int_0^1 \beta(a)n(t,a)da$ we find the renewal equation

$$y(t)=h(t)+\int_0^t K(s)y(t-s)ds ,$$

where $h(t)=\langle \beta, T_0^{\odot}(t)\psi\rangle =\int_t^1 \beta(a)\psi(a-t)da$. Note that $y(t)$ is the rate at which individuals are born at time t. Once y is known, $n(t)=T^{\odot}(t)\psi$ can be computed in the following way:

$$T^{\odot}(t)\psi = T_0^{\odot}(t)\psi + \int_0^t T_0^*(t-s)y(s)\delta ds ,$$

from which we get

$$n(t,a)=\begin{cases}\psi(a-t), & a \geqq t\\ y(t-a), & a<t .\end{cases}$$

Remark. Instead of $X=C_0[0,1]=\{\phi \in C[0,1]: \phi(1)=0\}$, we might also represent X by $C_0[0,1)$, the space of all continuous functions on $[0,1)$, which tend to zero as $a \to 1$. Then we would get $X^*=M[0,1)$. Of course, the difference in representation does not affect the results.

We emphasize that the computations above show that the functional analytic approach developed in this paper is in fact identical to the standard direct approach for the solution of age-dependent population problems via the renewal equation. The profit of the abstract reformulation is that items like linearized stability and Hopf bifurcation can now be handled in the standard way without any recourse to ad hoc arguments.

7. Some Remarks on Work in Progress

The basic assumption that X is $\odot$-reflexive with respect to A_0 (assumption 4.1) is quite restrictive. The following important example illustrates this.

Let $X = C_0(\mathbb{R}_+)$, i.e. the Banach space consisting of all continuous functions $\phi: \mathbb{R}_+ \to \mathbb{R}$ which vanish at infinity. Let $T(t)$ be the (strongly continuous) semigroup of translations on X,

$$(T(t)\phi)(x) = \phi(x+t) \ .$$

It is known [2] that $X^* = M(\mathbb{R}_+)$, the space of Borel measures on $\mathbb{R}_+$, $X^\odot = L^1(\mathbb{R}_+)$, $X^{\odot *} = L^\infty(\mathbb{R}_+)$ and $X^{\odot\odot} = BUC(\mathbb{R}_+) \neq C_0(\mathbb{R}_+) = X$. Here $BUC(\mathbb{R}_+)$ is the space of bounded uniformly continuous functions on $\mathbb{R}_+$, equipped with the supremum norm. Hence the condition of $\odot$-reflexivity is *not* satisfied.

In a forthcoming paper we develop a perturbation theory for the general case. One of the main steps there concerns the definition of the canonical duality pairing between elements of $X^{\odot\odot}$ and elements of X^*. Now, if $T_0(t)$ is a C_0-semigroup on X and B is a bounded linear perturbation from X to $X^{\odot *}$, then we construct the perturbed semigroup $T^{\odot\odot}(t)$ on $X^{\odot\odot}$ by solving the variation-of-constants formula

$$T^{\odot\odot}(t)x^{\odot\odot} = T_0^{\odot\odot}(t)x^{\odot\odot} + \int_0^t T_0^{\odot *}(t-s)BT^{\odot\odot}(s)x^{\odot\odot}\,ds \ ,$$

where B also denotes the canonical extension to $X^{\odot\odot}$. Alternatively, one could start with the (restricted) dual semigroup $T_0^\odot(t)$ on $X^\odot$, a bounded linear perturbation $C: X^\odot \to X^*$ and the variation-of-constants formula

$$T^\odot(t)x^\odot = T_0^\odot(t)x^\odot + \int_0^t T_0^*(t-s)CT^\odot(s)x^\odot\,ds \ ,$$

to construct a strongly continuous semigroup $T^\odot(t)$ on $X^\odot$. It turns out that both approaches are equivalent.

An important problem in semigroup theory concerns the behaviour for $t \to \infty$. In this respect, properties of the semigroup like the location of its spectrum, compactness and irreducibility (in the sense of positive operators) play a very important role, as indicated in [9]. In a forthcoming paper we shall deal extensively with such properties.

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References

1. Amann, H.: Dual semigroups and second order linear elliptic boundary value problems. Isr. J. Math. **45**, 225–254 (1983)
2. Butzer, P.L., Berens, H.: Semigroups of operators and approximations. Berlin, Heidelberg, New York: Springer 1967
3. Davies, E.B.: One-parameter semigroups. London: Academic Press 1980

4A. Desch, W., Schappacher, W.: On relatively bounded perturbations of linear C_0-semigroups. Ann. Sc. Norm. Super. Pisa Cl. Sci., IV. Ser. **11**, 327–341 (1984)
4B. Desch, W., Schappacher, W.: Spectral properties of finite-dimensional perturbed linear semigroups. J. Differ. Equations **59**, 80–102 (1985)
4C. Desch, W., Lasiecka, I., Schappacher, W.: Feedback boundary control problems for linear semigroups. Isr. J. Math. **51**, 177–207 (1985)
5. Diekmann, O.: Perturbed dual semigroups and delay equations. To appear in: Infinite dimensional dynamical systems. Proceedings, Lisbon, 1986
6. Greiner, G.: Perturbing the boundary conditions of a generator. Houston J. Math. (to appear)
7. Gyllenberg, M.: Stability of a nonlinear age-dependent population model containing a control variable. SIAM J. Appl. Math. **43**, 1418–1438 (1983)
8. Hale, J.K.: Theory of functional differential equations. Berlin, Heidelberg, New York: Springer 1977
9. Heijmans, H.J.A.M.: Dynamics of structured populations. Thesis, Univ. of Amsterdam, 1985
10. Hille, E., Phillips, R.S.: Functional analysis and semigroups. Providence, R.I.: Am. Math. Soc. 1957
11. Kappel, F., Schappacher, W.: Non-linear functional differential equations and abstract integral equations. Proc. Roy. Soc. Edinb. **84**A, 71–91 (1979)
12. Metz, J.A.J., Diekmann, O. (eds.): Dynamics of physiologically structured populations. Lecture Notes in Biomathematics 68. Berlin, Heidelberg, New York: Springer 1986
13. Pazy, A.: Semigroups of linear operators and applications to partial differential equations. Berlin, Heidelberg, New York: Springer 1983
14. Rudin, W.: Real and complex analysis, 2nd ed. New York: McGraw-Hill 1974
15. Suhadolc, A., Vidav, I.: Linearized Boltzmann equations in spaces of measures. Math. Balk. **3**, 514–529 (1973)
16. Thieme, H.R.: Well-posedness of physiologically structured population models for Daphnia magna. Preprint, CWI Report AM-R8609, 1986
17. Yosida, K.: Functional analysis. Berlin, Heidelberg, New York: Springer 1965

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