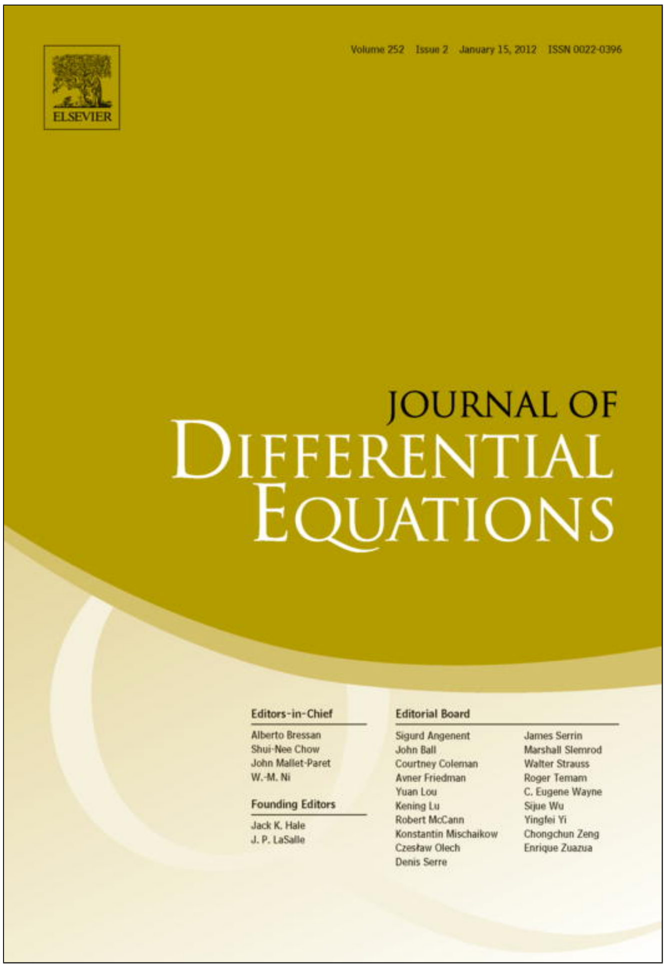
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Equations with infinite delay: Blending the abstract and the concrete

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ABSTRACT

Using perturbation theory for adjoint semigroups (a modification of sun-star calculus) we prove, in the case of infinite delay, the principle of linearized stability for nonlinear renewal equations, delay-differential equations and coupled systems of these two types of equations. Our results extend those of Diekmann et al. (1995) [13] and Diekmann et al. (2007) [14] to the case of infinite delay.

1. Introduction

Since the early 1970s semigroups of linear operators have played a major role in the analysis of retarded functional differential equations, also called delay-differential equations,

$$\dot{y}(t) = F(y_t), \quad t > 0, \tag{1.1}$$

with initial condition

$$y(\theta) = \psi(\theta), \quad \theta \leqslant 0. \tag{1.2}$$

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Here y_t denotes the history of y, that is,

$$y_t(\theta) = y(t+\theta) \quad \text{for } \theta \leqslant 0.$$

This has to a large extent been due to the influence of the books [22,23] by Jack Hale (see also [24]). Semigroups appear already in a useful reformulation of the problem (1.1)–(1.2). Let $T_0(t)$ act on a space of functions defined on $\mathbb{R}_-$ and be defined as translation to the left and extension by the value at zero. Then the original problem (1.1)–(1.2) can at least formally be written as an abstract integral equation of the variation-of-constants type for $u(t) = y_t$:

$$u(t) = T_0(t)\psi + \int_0^t T_0(t-s)HF\bigl(u(s)\bigr)\,ds, \tag{1.3}$$

where H is the Heaviside function

$$H(\theta) = \begin{cases} 0, & \theta < 0, \\ I, & \theta = 0. \end{cases}$$

The problem with (1.3) is that whatever function space one chooses as state space, the Heaviside function will typically not belong to it. On the other hand, translation to the left and extension by the value at zero can be applied to H so the formal approach works.

Using adjoint semigroups the variation-of-constants formula was given a precise functional analytical meaning in [4] and subsequently applied to delay-differential equations in [9,13]. The formal variation-of-constants formula (1.3) is replaced by the abstract integral equation

$$u(t) = T_0(t)\psi + j^{-1}\int_0^t T_0^{\odot *}(t-s)G\bigl(u(s)\bigr)\,ds. \tag{1.4}$$

The semigroup $T_0^{\odot *}$ is defined by a general procedure that we now describe.

Start with a strongly continuous semigroup T on a Banach space X. Its adjoint semigroup T^* is defined on the normed dual X^* of X. It is not necessarily strongly continuous on X^*. Denote by $X^\odot$ the maximal subspace of X^* on which T^* is strongly continuous. T^* leaves $X^\odot$ invariant and its restriction $T^\odot$ to $X^\odot$ is clearly a strongly continuous semigroup. Repeating the procedure of taking adjoints and restrictions to the space of strong continuity one obtains the semigroups $T^{\odot *}$ and $T^{\odot\odot}$ defined on the spaces $X^{\odot *}$ and $X^{\odot\odot}$, respectively. Next we define an embedding $j : X \to X^{\odot *}$ by

$$(j\psi)\bigl(\psi^\odot\bigr) = \psi^\odot(\psi), \quad \psi \in X,\ \psi^\odot \in X^\odot.$$

One has $j(X) \subset X^{\odot\odot}$ and $T^{\odot *}(t)j = jT(t)$ for $t \geqslant 0$. If $j(X) = X^{\odot\odot}$, then X is called *sun-reflexive* with respect to T.

This procedure gives a functional analytical method to extend the semigroup T_0 from X to the bigger space $X^{\odot *}$. G is a mapping from X to $X^{\odot *}$.

The integral in (1.4) is a weak*-Riemann integral and yields a well-defined element of $X^{\odot *}$ whenever $s \mapsto G(u(s))$ is continuous from $[0,t]$ to $X^{\odot *}$ equipped with the weak*-topology. If $s \mapsto G(u(s))$ is norm-continuous, then the integral actually belongs to $X^{\odot\odot}$ [4]. It follows that if X is sun-reflexive, then the abstract integral equation (1.4) makes sense as an equation for the unknown $u : \mathbb{R}_+ \to X$.

Perturbation theory for adjoint semigroups, as developed in the series [4–7,12] of papers, turned out to be a very powerful tool in the local stability and bifurcation theory of delay-differential equations (DDE) [13]. Later we showed [14] that the theory can also be applied to analyze delay equations not involving any derivative, that is, equations of the type

$$x(t) = F(x_t), \tag{1.5}$$

which one might also call nonlinear renewal equations. The only difference between the treatment of (1.1) and (1.5) is in the choice of the state space X, the unperturbed semigroup T_0 and the nonlinear perturbation G.

The abstract problems related to the concrete delay-differential equation (1.1) and renewal equation (1.5) are sun-reflexive if and only if the delay is finite. This is the reason why the book [13] and the paper [14] are restricted to equations with finite delay. When the delay is infinite it may not even be possible to find a concrete representation of the spaces $X^\odot$ and $X^{\odot *}$, so the whole theory based on (1.4) seems to fail.

It is the purpose of this paper to show that a slight modification of the framework is all that is needed to treat equations with infinite delay. Instead of the space $X^\odot$, which we cannot in general characterize, we choose a closed subspace X' of $X^\odot$ which separates points of X and which has a representation as a function space. In applications to (1.1) or (1.5) the nonlinear perturbation has finite dimensional range and one can by direct verification show that the integral in (1.4) belongs to $j(X)$. This has been done in somewhat greater generality for renewal equations in [10], but for delay-differential equations it has not been published before.

Lack of compactness is a second complication that arises when one moves from finite to infinite delay. When the delay h is finite, the unperturbed semigroup $T_0(t)$ will have finite dimensional range for $t > h$. In applications to renewal equations it is even nilpotent. In any case, T_0 is eventually compact when the delay is finite. Combined with the fact that G has finite dimensional range, this yields that the perturbed semigroup T for the linearized equation is eventually compact. As a consequence, the spectrum of the infinitesimal generator A of T is a pure point spectrum and the growth bound $\omega(T)$ of T equals the spectral bound $s(A)$ of A, which is the supremum of the real part of the eigenvalues of A. It follows that to determine the stability or instability of steady-states it suffices to locate the eigenvalues of A in the complex plane.

When the delay is infinite, this is no longer necessarily true and the inequality $s(A) \leqslant \omega(T)$ may be strict. Therefore the location of the eigenvalues of A does not tell the whole story. But here we switch from the abstract setting to the concrete delay equations. The linearized versions of the delay equations (1.1) and (1.5) are Volterra integro-differential equations or Volterra integral equations of the type

$$\dot{y}(t) = \int_{\mathbb{R}_+} \mu(ds)\, y(t-s), \tag{1.6}$$

$$x(t) = \int_0^\infty k(s)x(t-s)\, ds, \tag{1.7}$$

respectively. There is a large arsenal of tools for determining the asymptotic behavior of the solutions $x(t)$ and $y(t)$ to the Volterra equations (1.6) and (1.7) (see [19]) that can be used to determine the growth bound of the linearized semigroup T. This of course requires that we know that the concrete and the abstract equations are equivalent in a well-defined sense.

The *leitmotiv* of this paper is the blending of the abstract and the concrete. Once the equivalence of the concrete delay equation and the abstract integral equation has been proved, we can freely switch between the two frameworks according to our needs. Thus, for instance, existence and uniqueness of solutions is proved in the abstract framework as this covers equations of type (1.6), of type (1.7) and of mixed type in one go. Linearization around a steady-state is done in both frameworks and it is shown that the eigenvalues of the infinitesimal generator of the linearized semigroup correspond to zeros of a characteristic equation which is defined in terms of the Laplace transform of the kernel of the linearized concrete delay equation. The principle of linearized stability, which says that the stability of steady-states of the nonlinear equation is determined by the stability of zero for the linearized system, is proved in the abstract framework.

It should be noted that A. Grabosch in [17] considers (1.5) for functions with values in a Banach space. Her results, in particular Proposition 3.15 and Theorem 4.4, overlap with ours, but focus on stability. In fact, our results on instability appear to be new.

The paper is organized as follows. In Section 2 we present the abstract framework and prove that the Fréchet derivative at a steady-state of the solution semigroup of the nonlinear abstract integral equation is the solution semigroup of the corresponding linearized abstract integral equation. We also recall an abstract version of the principle of linearized stability due to Desch and Schappacher [8]. In Section 3 we show the equivalence of the nonlinear renewal equation and the abstract integral equation and prove the principle of linearized stability for renewal equations. This section generalizes the results of [14] to the case of infinite delay. In Section 4 we do the same for delay-differential equations. The results on linearized stability in [13] are thus generalized to the case of infinite delay. Many models of structured populations lead to systems of renewal equations coupled with delay-differential equations [15,20,21]. Therefore we give in Section 5 a short account of such coupled systems. As this is a straightforward combination of the results of Sections 3 and 4 we only state the results and do not present the proofs. In [14] we considered a structured metapopulation model and in [15] a model of size structured water fleas. In [11] we proved the instability part of the principle of linearized stability for the model given in [21]. All three models were of “mixed” type. In all cases the natural setting would have been a model with infinite delay, but we considered only finite delays in order to be able to use published results. We announced that the results could be extended to the case of infinite delay. Section 5 proves that these announcements were correct. In Section 6 we give some concluding remarks.

Note on terminology: In our paper [14] we called the undifferentiated equation (1.5) a *delay equation* or *Volterra functional equation* as opposed to the *delay-differential equation* (1.1). This terminology caused some confusion. On the one hand, the term “delay equation” has become an established name for the differentiated equation (1.1) and on the other hand both differentiated and undifferentiated equations with a nonanticipatory functional on the right-hand side are commonly called Volterra equations. In this paper we therefore use the term “delay-differential equation” for (1.1) and (*nonlinear*) *renewal equation* for (1.5). We refer to either of these equations, and to mixed systems as well, by the term “delay equation”. Traditionally the term “renewal equation” refers to the linear Volterra integral equation (1.7) for nonnegative kernels k in the critical case $\int_0^\infty k(t)\,dt = 1$. However, in many applications, for instance to population dynamics, Eq. (1.5) indeed describes renewal (of individuals, in particular) so we find the terminology appropriate.

2. The abstract setting

Let X be a Banach space and let T_0 be a C_0-semigroup on X with infinitesimal generator A_0. Let X' be a closed subspace of $X^\odot$, which is invariant under $T_0^\odot$ and which separates points of X. We denote the restriction of $T_0^\odot$ to X' by T_0'. It is a C_0-semigroup on X' and its infinitesimal generator A_0' is the part of $A_0^\odot$ in X'.

The duality pairing between a space and its dual is denoted by $\langle\,,\rangle$. It is convenient to choose the order of the arguments in the pairing such that elements of X and X'^* are to the left and elements of X^* and X' are to the right. We thus for instance write

$$\varphi'(\varphi) = \langle \varphi, \varphi' \rangle \quad \text{but} \quad \varphi'^*(\varphi') = \langle \varphi'^*, \varphi' \rangle \quad \text{for } \varphi \in X,\ \varphi' \in X',\ \varphi'^* \in X'^*.$$

With this convention we define the linear mapping $j : X \to X'^*$ by

$$\langle jx, x' \rangle = \langle x, x' \rangle, \quad x \in X,\ x' \in X'. \tag{2.1}$$

The mapping j is obviously bounded and it is an injection because X' separates points of X.

We consider the abstract integral equation

$$u(t) = T_0(t)\varphi + j^{-1}\int_0^t T_0'^*(t-s)(\ell \circ F)\big(u(s)\big)\,ds, \quad t \geqslant 0, \tag{AIE}$$

where $\ell : \mathbb{R}^m \to X'^*$ is a given bounded linear injection and $F : X \to \mathbb{R}^m$ is a given nonlinear mapping. In order for the equation to make sense, the integral on the right-hand side of (AIE) must belong to the range of j and to prove existence and uniqueness using Banach's fixed point theorem (and to subsequently do stability analysis) we need an estimate of the integral. We therefore make the following assumption.

Hypothesis 2.1. There exist an $M \geqslant 1$ and an $\omega \in \mathbb{R}$ such that for all continuous functions $h : \mathbb{R} \to \mathbb{R}^m$ and all $t > 0$ one has

$$\int_0^t T_0'^*(t-s)\ell h(s)\,ds \in j(X), \tag{2.2}$$

$$\left\| j^{-1}\int_0^t T_0'^*(t-s)\ell h(s)\,ds \right\| \leqslant M\int_0^t e^{\omega(t-s)}\big|h(s)\big|\,ds. \tag{2.3}$$

The point of Hypothesis 2.1 is that in many applications in which sun-reflexivity does not hold, e.g. delay equations with infinite delay, the mapping ℓ is determined by the problem and the hypothesis is easy to verify directly.

Theorem 2.2. *Let Hypothesis* 2.1 *hold and assume that F is* (*globally*) *Lipschitz continuous. Then for all $\varphi \in X$, the abstract integral equation* (AIE) *has a unique solution $u(t) = \Sigma(t)\varphi$ on* $[0, \infty)$. *The family $\{\Sigma(t)\}_{t\geqslant 0}$ of nonlinear operators is a semigroup on X.*

Proof. In the sun-reflexive case we may take $X' = X^\odot$ and then $j(X) = X^{\odot\odot}$ so that the range condition (2.2) and the estimate (2.3) of Hypothesis 2.1 always hold [4]. The existence and uniqueness proof in the sun-reflexive case [6] uses sun-reflexivity *only* to ensure (2.2) and (2.3). Thus that proof carries over *verbatim* to the present case. $\square$

A *steady-state* of the dynamical system (semigroup) Σ is an element $\overline{\varphi} \in X$ for which $\Sigma(t)\overline{\varphi} = \overline{\varphi}$ for all $t \geqslant 0$. Our next theorem states that if F is sufficiently smooth, then $\Sigma(t)$ is differentiable at $\overline{\varphi}$ and its derivative is a linear semigroup satisfying a linear abstract integral equation.

Theorem 2.3. *Let F be continuously Fréchet differentiable and assume that $\overline{\varphi}$ is a steady-state of Σ. Then the nonlinear operator $\Sigma(t)$ has a Fréchet derivative $T(t) = D\Sigma(t)(\overline{\varphi})$ at $\overline{\varphi}$, uniformly on compact t-intervals. For every $\varphi \in X$, $T(t)\varphi$ is the unique solution of the linear abstract integral equation*

$$T(t)\varphi = T_0(t)\varphi + j^{-1}\int_0^t T_0'^*(t-s)\ell DF(\overline{\varphi})T(s)\varphi\,ds. \tag{LAIE}$$

Note: In the proof of this theorem, as well as in the rest of the paper, we use the letter M to denote an *a priori* finite bound, the value of which may change from line to line.

Proof of Theorem 2.3. Without loss of generality we can take $\overline{\varphi} = 0$ (see e.g. [13, p. 222]). Note that because ℓ is an injection, this forces $F(0) = 0$.

Subtracting (LAIE) from (AIE) one obtains

$$\begin{aligned}\Sigma(t)\varphi - T(t)\varphi = j^{-1}\int_0^t T_0'^*(t-s)\ell\big(F\big(\Sigma(s)\varphi\big) - DF(0)\Sigma(s)\varphi\big)\,ds \\ + j^{-1}\int_0^t T_0'^*(t-s)\ell DF(0)\big(\Sigma(s)\varphi - T(s)\varphi\big)\,ds.\end{aligned}$$

Using the estimate (2.3) it follows that

$$u(t) \leqslant M\int_0^t e^{\omega(t-s)} f(s)\,ds + M\int_0^t e^{\omega(t-s)}\big\|DF(0)\big\|u(s)\,ds, \tag{2.4}$$

where

$$\begin{aligned}u(t) &= \big\|\Sigma(t)\varphi - T(t)\varphi\big\|, \\ f(t) &= \big|F\big(\Sigma(t)\varphi\big) - DF(0)\Sigma(t)\varphi\big|.\end{aligned}$$

Multiplying (2.4) by $e^{-\omega t}$ one obtains

$$v(t) \leqslant M\int_0^t e^{-\omega s} f(s)\,ds + M\int_0^t \big\|DF(0)\big\|v(s)\,ds, \tag{2.5}$$

where

$$v(t) = e^{-\omega t}u(t).$$

Now (the generalized) Gronwall's lemma gives that $v(t) \leqslant w(t)$, where w is the solution of the comparison equation

$$w(t) = M\int_0^t e^{-\omega s} f(s)\,ds + M\int_0^t \big\|DF(0)\big\|w(s)\,ds,$$

that is,

$$w(t) = M\int_0^t e^{M\|DF(0)\|(t-s)}e^{-\omega s} f(s)\,ds.$$

Hence

$$u(t) = e^{\omega t}v(t) \leqslant e^{\omega t}w(t) = M\int_0^t e^{(M\|DF(0)\|+\omega)(t-s)} f(s)\,ds,$$

from which it follows that

$$\left\| \Sigma(t)\varphi - T(t)\varphi \right\| \leqslant M \sup_{0\leqslant s\leqslant t} \left| F\left(\Sigma(s)\varphi\right) - DF(0)\Sigma(s)\varphi \right|, \tag{2.6}$$

where the finite bound M depends on t.

An application of Gronwall's lemma, similar to the one above, gives that

$$\left\| \Sigma(t)\varphi \right\| \leqslant M\|\varphi\|$$

for all $\varphi \in X$. Because F is continuously Fréchet differentiable (and $F(0) = 0$) we have

$$\left| F\left(\Sigma(s)\varphi\right) - DF(0)\Sigma(s)\varphi \right| \leqslant M\|\varphi\|\varepsilon\left(\left\|\Sigma(s)\varphi\right\|\right), \tag{2.7}$$

where ε is a function tending to zero as its argument goes to zero. (2.6) and (2.7) now yield the desired conclusion $T(t) = D\Sigma(t)(\overline{\varphi})$. $\square$

Theorem 2.3 is instrumental in proving the *principle of linearized stability* for Σ because it provides the basic assumption of the following theorem originally due to Desch and Schappacher.

Theorem 2.4. *(See [8], [6, Theorems 4.2 and 4.3], [13, Corollary 5.12].) Let Σ be a strongly continuous nonlinear semigroup. Let $\overline{\varphi}$ be a steady-state of Σ and assume that for each $t \geqslant 0$, $\Sigma(t)$ has a (uniform) Fréchet derivative $T(t)$ at $\overline{\varphi}$. Let A be the infinitesimal generator of T. Assume further that X admits a decomposition*

$$X = X_- \oplus X_+$$

into two $T(t)$-invariant subspaces X_- and X_+ such that

(i) *X_+ is finite dimensional,*
(ii) *the restriction of $T(t)$ to X_- converges exponentially to 0 as $t \to \infty$.*

Then $\overline{\varphi}$ is

(a) *(locally) exponentially stable if $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A|_{X_+})$,*
(b) *unstable if there exists a $\lambda \in \sigma(A|_{X_+})$ with $\operatorname{Re}\lambda > 0$.*

When the semigroup T_0 is eventually compact and the nonlinear perturbation is compact (so in particular when $\ell \circ F$ has finite dimensional range), Theorem 2.4 is particularly easy to apply. The reason is that in that case the spectrum of the infinitesimal generator A of the linearized semigroup T is a pure point spectrum and the growth bound (i.e., the infimum of all $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant Me^{\omega t}$ for some $M \geqslant 1$) equals the spectral bound (the supremum of the real part of all spectral values), see, e.g., [1, Theorem 2.1, p. 209]. Eventual compactness holds e.g. in applications to equations with finite delay. This allowed us in our treatise [14] of delay equations with finite delay to stay abstract: We characterized the eigenvalues of A as the roots of a *characteristic equation* and concluded from Theorem 2.4 that if all the eigenvalues lie in the left half-plane, then $\overline{\varphi}$ is exponentially stable, whereas it is unstable if there is at least one eigenvalue in the right half-plane.

In applications to equations with infinite delay eventual compactness is lost and we cannot deduce the growth bound from the spectral bound. However, as we shall see in the following sections, the growth bound can be determined from the concrete equation using the Paley–Wiener theorem.

3. Renewal equations

We consider the (concrete) initial value problem

$$x(t) = F(x_t), \quad t > 0, \tag{RE}$$

$$x(\theta) = \varphi(\theta), \quad \theta \in (-\infty, 0], \tag{IC}$$

consisting of a nonlinear renewal equation (RE) specifying the rule for extending the unknown function x towards the future on the basis of the history given by the initial condition (IC). Here the given function F and the unknown function x take on values in $\mathbb{R}^m$ and x_t denotes for each $t \geqslant 0$ the translated function defined by

$$x_t(\theta) := x(t+\theta), \quad -\infty < t \leqslant 0. \tag{3.1}$$

Let $\varrho > 0$. We require the initial condition φ to belong to the state space (history space) $X = L^1_\varrho(\mathbb{R}_-; \mathbb{R}^m)$ of absolutely integrable (with respect to the weight function $\theta \mapsto e^{\varrho\theta}$) functions on $\mathbb{R}_-$. To be precise, $X = L^1_\varrho(\mathbb{R}_-; \mathbb{R}^m)$ consists of all (equivalence classes of) measurable functions $\varphi : \mathbb{R}_- = (-\infty, 0] \to \mathbb{R}^m$ such that the weighted integral

$$\|\varphi\|_{1,\varrho} = \int_{-\infty}^{0} e^{\varrho\theta} \left|\varphi(\theta)\right| d\theta \tag{3.2}$$

is finite. $\|\varphi\|_{1,\varrho}$ is of course the norm of φ in $L^1_\varrho(\mathbb{R}_-; \mathbb{R}^m)$. A first reason for the weight is that we want to consider *steady-states,* that is, constant solutions of (RE), and constants do not belong to L^1 when the domain (delay) is infinite. The reason for not choosing a space of continuous functions as state space will become clear in a minute. The second reason for the weight is that we want some Laplace transforms to be defined in a strip to the left of the imaginary axis (see for instance Proposition 3.8).

So when we say that F maps X into $\mathbb{R}^m$, it means that an equivalence class is mapped to an m-vector. In concrete cases one often first defines a map that assigns an m-vector to a function and next checks that if we consider a different function belonging to the same equivalence class, the same m-vector is assigned. The second step of checking is often left implicit.

Note that the solution x of (RE) is, for $t \geqslant 0$, defined pointwise and not just almost everywhere. Moreover, if F is continuous, so is x for $t \geqslant 0$, simply because translation is continuous in L^1.

In the previous section we linearized the abstract integral equation (AIE) and showed that the Fréchet derivative at a steady-state of the nonlinear semigroup associated with (AIE) is a linear semigroup satisfying the linear abstract integral equation (LAIE). Next we linearize the concrete nonlinear renewal equation (RE).

Let $\bar{x}$ be a steady-state, that is, a constant solution, of (RE). Then

$$\bar{x} = F(\bar{x}),$$

where we abuse notation and use the same symbol $\bar{x}$ to denote both a constant function (an element of X) and the constant value (an element of $\mathbb{R}^m$) it takes on. We assume that $F : X \to \mathbb{R}^m$ is continuously Fréchet differentiable, expand $F(\bar{x} + x_t)$ about $\bar{x}$, neglect higher order terms and arrive at the linearization

$$x(t) = DF(\bar{x})x_t. \tag{3.3}$$

By the definition of the Fréchet derivative, $DF(\bar{x})$ is a bounded linear operator from $X = L^1_\varrho(\mathbb{R}_-;\mathbb{R}^m)$ into $\mathbb{R}^m$. By Riesz' representation theorem, there exists a unique $k \in L^\infty_\varrho(\mathbb{R}_+;\mathbb{R}^{m\times m}) = \{k : \mathbb{R}_+ \to \mathbb{R}^{m\times m}\colon \sup_{s\in\mathbb{R}_+} |e^{\varrho s}k(s)| < \infty\}$ such that

$$DF(\bar{x})\varphi = \int_0^\infty k(s)\varphi(-s)\,ds. \tag{3.4}$$

Combining (3.3) and (3.4) one obtains the linear renewal equation

$$x(t) = \int_0^\infty k(s)x(t-s)\,ds, \quad t > 0. \tag{LRE}$$

Sometimes we write (LRE) as

$$x(t) = \langle x_t, k\rangle, \tag{LRE}$$

where the pairing between an m-dimensional column vector valued function φ defined on $\mathbb{R}_-$ and an $m \times m$-matrix valued function k defined on $\mathbb{R}_+$ is given by

$$\langle \varphi, k\rangle := \int_0^\infty k(s)\varphi(-s)\,ds. \tag{3.5}$$

The Riesz representation theorem ensured that $k \in L^\infty_\varrho(\mathbb{R}_+;\mathbb{R}^{m\times m})$. Later in this section we shall apply the Paley–Wiener theorem to give a criterion that guarantees that the resolvent kernel r of k belongs to L^1_ϱ. This requires k to be in L^1_ϱ. The flexibility in the choice of the value of ϱ gives us sufficient leeway to make this condition hold. The following lemma is obvious.

Lemma 3.1. *Let* $0 < \varrho_1 < \varrho_2$. *Then*

$$L^1_{\varrho_1}\big(\mathbb{R}_-;\mathbb{R}^m\big) \subset L^1_{\varrho_2}\big(\mathbb{R}_-;\mathbb{R}^m\big) \tag{3.6}$$

and

$$\|\varphi\|_{1,\varrho_2} \leqslant \|\varphi\|_{1,\varrho_1} \tag{3.7}$$

for $\varphi \in L^1_{\varrho_1}(\mathbb{R}_-;\mathbb{R}^m)$.

Corollary 3.2. *Let* $0 < \varrho_1 < \varrho_2$. *Let* $F : L^1_{\varrho_2}(\mathbb{R}_-;\mathbb{R}^m) \to \mathbb{R}^m$ *be Fréchet differentiable at* $\overline{\varphi} \in L^1_{\varrho_1}(\mathbb{R}_-;\mathbb{R}^m) \subset L^1_{\varrho_2}(\mathbb{R}_-;\mathbb{R}^m)$ *with Fréchet derivative* $DF(\overline{\varphi})$ *represented by the kernel* $k \in L^\infty_{\varrho_2}(\mathbb{R}_+;\mathbb{R}^{m\times m})$. *Then* F *is Fréchet differentiable at* $\overline{\varphi}$ *as a function from* $L^1_{\varrho_1}(\mathbb{R}_-;\mathbb{R}^m)$ *into* $\mathbb{R}^m$. *Its Fréchet derivative is represented by* k *and* k *belongs not only (by definition) to* $L^\infty_{\varrho_1}(\mathbb{R}_+;\mathbb{R}^{m\times m})$ *but also to* $L^1_{\varrho_1}(\mathbb{R}_+;\mathbb{R}^{m\times m})$.

Proof. Because F regarded as a function from $L^1_{\varrho_2}(\mathbb{R}_-;\mathbb{R}^m)$ into $\mathbb{R}^m$ is Fréchet differentiable at $\overline{\varphi}$, one has using Lemma 3.1

$$\big|F(\overline{\varphi}+\varphi) - F(\overline{\varphi}) - \langle\varphi, k\rangle\big| \leqslant \|\varphi\|_{1,\varrho_2}\varepsilon\big(\|\varphi\|_{1,\varrho_2}\big) \leqslant \|\varphi\|_{1,\varrho_1}\varepsilon\big(\|\varphi\|_{1,\varrho_2}\big), \tag{3.8}$$

where ε is a function tending to zero as its argument tends to zero. As $\|\varphi\|_{1,\varrho_1}$ tends to zero also $\|\varphi\|_{1,\varrho_2}$ tends to zero by (3.7) and hence (3.8) shows that F is Fréchet differentiable at $\overline{\varphi}$ regarded as function defined on $L^1_{\varrho_1}(\mathbb{R}_-;\mathbb{R}^m)$ and the derivative is represented by k. The inequality

$$\int_0^\infty e^{\varrho_1 s}\big|k(s)\big|\,ds = \int_0^\infty e^{(\varrho_1-\varrho_2)s}e^{\varrho_2 s}\big|k(s)\big|\,ds \leqslant \frac{1}{\varrho_2-\varrho_1}\sup_{s\in\mathbb{R}_+} e^{\varrho_2 s}\big|k(s)\big|$$

shows that $k\in L^1_{\varrho_1}(\mathbb{R}_+;\mathbb{R}^{m\times m})$. $\square$

From now on we shall always assume that ϱ is chosen such that $k\in L^1_{\varrho}(\mathbb{R}_+;\mathbb{R}^{m\times m})$.

Next we reformulate (RE)–(IC) as an abstract integral equation. In other words, we show that there is a one-to-one correspondence between solutions of (RE)–(IC) and solutions of (AIE) for certain choices of T_0 and ℓ. To this end, we consider on X the strongly continuous semigroup T_0 defined by translation and extension by zero:

$$\big(T_0(t)\varphi\big)(\theta) = \begin{cases} \varphi(t+\theta), & -\infty<\theta\leqslant -t,\\ 0, & -t<\theta\leqslant 0,\end{cases} \qquad \varphi\in X,\ t\geqslant 0. \tag{3.9}$$

Obviously T_0 does not leave the continuous functions invariant and that is the reason why we could not choose a space of continuous functions as state space.

As X' we choose the space of all functions $\varphi':\mathbb{R}\to\mathbb{R}^m$ such that $\theta\mapsto e^{\varrho\theta}\varphi'(\theta)$ is bounded and uniformly continuous with the norm

$$\big\|\varphi'\big\|_{\infty,\varrho} = \sup_{\theta\in\mathbb{R}_+} e^{\varrho\theta}\big\|\varphi'(\theta)\big\| < \infty \tag{3.10}$$

and the pairing

$$\big\langle\varphi,\varphi'\big\rangle = \int_{\mathbb{R}_+}\varphi(-\theta)\cdot\varphi'(\theta)\,d\theta, \tag{3.11}$$

where the dot denotes the usual inner product in $\mathbb{R}^m$. As a matter of fact, X' is (isometrically isomorphic to) $X^{\odot}$ (see [3,31] for the case without and [10] for the case with a weight) but this will play no role in what follows.

A word of warning: Here and in what follows φ' denotes a generic element of the space X' and *not* the derivative of φ.

The adjoint semigroup T_0' of T_0 with respect to the pairing (3.11) is again a family of shift operators:

$$\big(T_0'(t)\varphi'\big)(\theta) = \varphi'(t+\theta), \quad \theta\geqslant 0,\ \varphi'\in X',\ t\geqslant 0. \tag{3.12}$$

The semigroup T_0' is obviously strongly continuous on X'.

Because X' is a space of continuous functions, evaluation at zero is well defined. Therefore we can define a linear mapping $\ell:\mathbb{R}^m\to X'^*$ by

$$\big\langle\ell z,\varphi'\big\rangle = z\cdot\varphi'(0), \quad z\in\mathbb{R}^m,\ \varphi'\in X'. \tag{3.13}$$

We claim that with these choices of T_0 and ℓ, (RE)–(IC) are indeed equivalent to (AIE). This was proved in a more general context in [10] but for completeness we repeat it here. Next we prove that Hypothesis 2.1 holds.

Lemma 3.3. *For every $z \in \mathbb{R}^m$ and $\varphi' \in X'$ one has*

$$\langle T_0'^*(t)\ell z, \varphi' \rangle = z \cdot \varphi'(t), \quad t \geqslant 0.$$

Proof.

$$\langle T_0'^*(t)\ell z, \varphi' \rangle = \langle \ell z, T_0'(t)\varphi' \rangle = z \cdot \big(T_0'(t)\varphi'\big)(0) = z \cdot \varphi'(t). \qquad \square$$

Lemma 3.4. *Let $h : \mathbb{R}_+ \to \mathbb{R}^m$ be a continuous function. Then, for every $\varphi' \in X'$ one has*

$$\left\langle \int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau, \varphi' \right\rangle = \int_0^t h(t-\tau) \cdot \varphi'(\tau)\,d\tau, \quad t \geqslant 0.$$

Proof. Using Lemma 3.3 one gets

$$\begin{aligned} \left\langle \int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau, \varphi' \right\rangle &= \int_0^t \langle T_0'^*(t-\tau)\ell h(\tau), \varphi' \rangle\,d\tau = \int_0^t h(\tau) \cdot \varphi'(t-\tau)\,d\tau \\ &= \int_0^t h(t-\tau) \cdot \varphi'(\tau)\,d\tau. \qquad \square \end{aligned}$$

Corollary 3.5. *Let $h : \mathbb{R}_+ \to \mathbb{R}^m$ be a continuous function and define $\varphi \in X = L^1_\varrho(\mathbb{R}_-; \mathbb{R}^m)$ $(\varrho > 0)$ by*

$$\varphi(\theta) = \begin{cases} h(t+\theta), & -t \leqslant \theta \leqslant 0, \\ 0, & -\infty < \theta < -t. \end{cases} \tag{3.14}$$

Then

$$\int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau = j\varphi. \tag{3.15}$$

In particular, $\int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau \in j(X)$ and

$$\left\| j^{-1}\left(\int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau \right) \right\|_{1,\varrho} \leqslant \int_0^t e^{-\varrho(t-s)} |h(s)|\,ds, \quad t \geqslant 0, \tag{3.16}$$

that is, Hypothesis 2.1 *holds with $M = 1$ and $\omega = -\varrho$.*

Proof. For each $\varphi' \in X'$ we have by the definition of φ and Lemma 3.4:

$$\begin{aligned} \langle \varphi, \varphi' \rangle &= \int_{-\infty}^0 \varphi(\theta) \cdot \varphi'(-\theta)\,d\theta = \int_{-t}^0 h(t+\theta) \cdot \varphi'(-\theta)\,d\theta = \int_0^t h(t-\theta) \cdot \varphi'(\theta)\,d\theta \\ &= \left\langle \int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau, \varphi' \right\rangle. \end{aligned}$$

The definition (2.1) of the embedding $j : X \to X'^*$ now yields (3.15). The estimate (3.16) follows readily:

$$\left\| j^{-1}\left(\int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau\right)\right\|_{1,\varrho}$$
$$= \|\varphi\|_{1,\varrho} = \int_{-\infty}^0 e^{\varrho\theta}\left\|\varphi(\theta)\right\|d\theta = \int_{-t}^0 e^{\varrho\theta}\left|h(t+\theta)\right|d\theta = \int_0^t e^{-\varrho(t-\tau)}\left|h(\tau)\right|d\tau. \qquad \square$$

An application of Corollary 3.5 to the function $h = F \circ u$ for a continuous function $u : \mathbb{R}_+ \to X$ immediately gives the following result:

Corollary 3.6. *Let $u : \mathbb{R}_+ \to X$ be continuous. Then*

$$\left(j^{-1}\int_0^t T_0'^*(t-s)\ell F\big(u(s)\big)\,ds\right)(\theta) = \begin{cases} F(u(t+\theta)), & -t \leqslant \theta \leqslant 0, \\ 0, & -\infty < \theta < -t. \end{cases}$$

We are now ready to state and prove the equivalence of (RE)–(IC) and (AIE).

Theorem 3.7. *Let $\varphi \in X = L^1_\varrho(\mathbb{R}_-; \mathbb{R}^m)$ be given.*

(a) *Suppose that $x \in L^1_{\mathrm{loc}}((-\infty,\infty); \mathbb{R}^m)$ satisfies* (RE)–(IC). *Then the function $u : [0,\infty) \to X$ defined by $u(t) := x_t$ is continuous and satisfies* (AIE).
(b) *If $u : [0,\infty) \to X$ is continuous and satisfies* (AIE), *then the function x defined by*

$$x(t) := \begin{cases} \varphi(t) & \text{for } -\infty < t < 0, \\ F(u(t)) & \text{for } t \geqslant 0 \end{cases} \tag{3.17}$$

specifies an element of $L^1_{\mathrm{loc}}((-\infty,\infty); \mathbb{R}^m)$ and satisfies (RE)–(IC).

Proof. (a) The continuity of $u(t) = x_t$ follows from the continuity of translation in L^1. Fix $t \geqslant 0$. By the definition of T_0 one has for $-t \leqslant \theta \leqslant 0$

$$u(t)(\theta) - \big(T_0(t)\varphi\big)(\theta) = x(t+\theta) - 0 = F(x_{t+\theta}) = F\big(u(t+\theta)\big)$$

and for $-\infty < \theta < -t$

$$u(t)(\theta) - \big(T_0(t)\varphi\big)(\theta) = x(t+\theta) - \varphi(t+\theta) = \varphi(t+\theta) - \varphi(t+\theta) = 0.$$

Corollary 3.6 shows that in both cases $u(t)(\theta) - (T_0(t)\varphi)(\theta)$ equals $(j^{-1}\int_0^t T_0'^*(t-s)\ell F(u(s))\,ds)(\theta)$ and thus u satisfies (AIE).

(b) By definition (3.17), (IC) holds and for $t \geqslant 0$,

$$x(t) = F\big(u(t)\big). \tag{3.18}$$

It thus remains to be shown that $u(t) = x_t$. Corollary 3.6 and (3.18) give

$$x_t(\theta) = x(t+\theta) = F\big(u(t+\theta)\big) = \left(j^{-1}\int_0^t T_0'^{*}(t-s)\ell F\big(u(s)\big)\,ds\right)(\theta)$$
$$= u(t)(\theta) - \big(T_0(t)\varphi\big)(\theta) = u(t)(\theta)$$

for $-t < \theta \leqslant 0$ and

$$x_t(\theta) = x(t+\theta) = \varphi(t+\theta) = \big(T_0(t)\varphi\big)(\theta) = u(t)(\theta)$$

for $-\infty < \theta < -t$, so indeed $u(t) = x_t$. $\square$

Let us recapitulate the situation. We have on the one hand the concrete renewal equation

$$x(t) = F(x_t), \quad t > 0, \tag{RE}$$

with initial condition

$$x_0(\theta) = \varphi(\theta), \quad \theta \in (-\infty, 0], \tag{IC}$$

and on the other hand the abstract integral equation

$$u(t) = T_0(t)\varphi + j^{-1}\int_0^t T_0'^{*}(t-s)(\ell \circ F)\big(u(s)\big)\,ds. \tag{AIE}$$

By Theorem 3.7 we know that they are equivalent. As a special case we have for the linearization the equivalent systems

$$x(t) = \langle x_t, k\rangle, \quad t > 0, \tag{LRE}$$
$$x_0(\theta) = \varphi(\theta), \quad \theta \in (-\infty, 0], \tag{IC}$$

and

$$T(t)\varphi = T_0(t)\varphi + j^{-1}\int_0^t T_0'^{*}(t-s)\ell\big\langle T(s)\varphi, k\big\rangle\,ds, \tag{LAIE}$$

with the duality brackets defined by (3.5) and the equivalence given by

$$x(t) := \begin{cases} \varphi(t) & \text{for } -\infty < t < 0, \\ \langle T(t)\varphi, k\rangle & \text{for } t \geqslant 0 \end{cases} \tag{3.19}$$

and

$$T(t)\varphi = x_t. \tag{3.20}$$

Because of this equivalence we can freely switch between the abstract and the concrete, according to our needs.

In order to apply Theorem 2.4 we have to find an invariant direct sum decomposition $X = X_+ \oplus X_-$ of the state space X into a finite dimensional subspace X_+ and a subspace X_- on which the semigroup $T(t)$ tends uniformly exponentially to zero as $t \to \infty$. It turns out that the part of the spectrum of A that lies in the right half-plane consists of eigenvalues and that a proper choice of X_+ is the direct sum of the corresponding (generalized) eigenspaces. If there are no eigenvalues with positive real part we take X_+ to be the trivial subspace $\{0\}$.

By a well-known theorem (see, e.g., [32]) the resolvent $R(\lambda, A)\varphi = (\lambda I - A)^{-1}\varphi$ is the Laplace transform of $T(t)\varphi$ for all λ with real part larger than the abscissa of convergence:

$$R(\lambda, A)\varphi = \big(\widehat{T(\cdot)\varphi}\big)(\lambda) = \int_0^\infty e^{-\lambda t} T(t)\varphi\, dt. \tag{3.21}$$

From (3.19), (3.20) and (3.21) we get

$$\begin{aligned}\big(R(\lambda, A)\varphi\big)(\theta) &= \int_0^\infty e^{-\lambda t} x_t(\theta)\, dt = \int_0^\infty e^{-\lambda t} x(t+\theta)\, dt = \int_\theta^\infty e^{-\lambda(\sigma-\theta)} x(\sigma)\, d\sigma \\ &= e^{\lambda\theta} \int_\theta^\infty e^{-\lambda\sigma} x(\sigma)\, d\sigma = e^{\lambda\theta}\left(\widehat{x}(\lambda) + \int_\theta^0 e^{-\lambda\sigma}\varphi(\sigma)\, d\sigma\right).\end{aligned} \tag{3.22}$$

By (3.19), (LRE) can be written as

$$\begin{aligned} x(t) &= \int_0^t k(s)x(t-s)\, ds + \int_t^\infty k(s)\varphi(t-s)\, ds \\ &= \int_0^t k(s)x(t-s)\, ds + \int_{-\infty}^0 k(t-s)\varphi(s)\, ds. \end{aligned} \tag{3.23}$$

Taking the Laplace transform of (3.23) one obtains

$$\widehat{x}(\lambda) = \widehat{k}(\lambda)\widehat{x}(\lambda) + Q(\lambda)\varphi, \tag{3.24}$$

where

$$Q(\lambda)\varphi = \int_0^\infty e^{-\lambda t} \int_{-\infty}^0 k(t-s)\varphi(s)\, ds\, dt. \tag{3.25}$$

It follows from (3.24) that

$$\widehat{x}(\lambda) = \big(I - \widehat{k}(\lambda)\big)^{-1} Q(\lambda)\varphi. \tag{3.26}$$

Substituting the expression (3.26) for $\widehat{x}(\lambda)$ into (3.22), one obtains

$$\big(R(\lambda, A)\varphi\big)(\theta) = e^{\lambda\theta}\left(\big(I - \widehat{k}(\lambda)\big)^{-1} Q(\lambda)\varphi + \int_\theta^0 e^{-\lambda s}\varphi(s)\, ds\right). \tag{3.27}$$

Thus the resolvent $R(\lambda, A)$ exists as a bounded linear operator from X into X provided the right-hand side of (3.27), as a function of θ, is a well-defined element of X and depends continuously on φ.

Let us denote the second term on the right-hand side of (3.27) by $(H(\lambda)\varphi)(\theta)$, that is,

$$\big(H(\lambda)\varphi\big)(\theta) = e^{\lambda\theta} \int_\theta^0 e^{-\lambda s}\varphi(s)\,ds.$$

Then

$$\begin{aligned}\big\|H(\lambda)\varphi\big\|_{1,\varrho} &\leqslant \int_{-\infty}^0 e^{\varrho\theta} e^{\operatorname{Re}\lambda\theta} \int_\theta^0 e^{-\operatorname{Re}\lambda s}\big|\varphi(s)\big|\,ds\,d\theta = \int_{-\infty}^0 \int_{-\infty}^s e^{(\varrho+\operatorname{Re}\lambda)\theta}\,d\theta\, e^{-\operatorname{Re}\lambda s}\big|\varphi(s)\big|\,ds \\ &= \int_{-\infty}^0 \frac{1}{\varrho+\operatorname{Re}\lambda} e^{(\varrho+\operatorname{Re}\lambda)s} e^{-\operatorname{Re}\lambda s}\big|\varphi(s)\big|\,ds = \frac{1}{\varrho+\operatorname{Re}\lambda}\|\varphi\|_{1,\varrho}.\end{aligned}$$

We conclude that for all complex λ with $\operatorname{Re}\lambda > -\varrho$, $H(\lambda)$ is a bounded linear operator from X into X.

We now turn to the first term on the right-hand side of (3.27). First of all, this term is not defined if $I - \widehat{k}(\lambda)$ is not invertible, which happens precisely if λ is a root of the *characteristic equation*

$$\det\big(I - \widehat{k}(\lambda)\big) = 0. \tag{3.28}$$

By a standard result in complex function theory, the roots of (3.28) are isolated points in $\mathbb{C}$. Let λ be a root of (3.28). Direct verification shows that $x(t) = e^{\lambda t}\gamma$, $t \in \mathbb{R}$, is a solution of (LRE)

$$x(t) = \int_0^\infty k(s)x(t-s)\,ds$$

if and only if γ is an eigenvector of $\widehat{k}(\lambda)$ corresponding to the eigenvalue 1. It follows that φ defined by

$$\varphi(\theta) = e^{\lambda\theta}\gamma, \quad \theta \leqslant 0,$$

satisfies

$$T(t)\varphi = e^{\lambda t}\varphi, \quad t \geqslant 0,$$

and hence that λ is an eigenvalue of A with corresponding eigenvector φ. The set of roots of (3.28) (the *characteristic roots*) in the half-plane $\operatorname{Re}\lambda > -\varrho$ is therefore a subset of the *point spectrum* of A. Next we show that there are no other spectral values in the half-plane $\operatorname{Re}\lambda > -\varrho$.

Let

$$\big(M(\lambda)\varphi\big)(\theta) = e^{\lambda\theta} Q(\lambda)\varphi. \tag{3.29}$$

If λ is not a characteristic root, then the first term on the right-hand side of (3.27) defines a bounded linear operator from X into X if and only if $M(\lambda)$ is a bounded linear operator from X into X. The inequality

$$\left\|M(\lambda)\varphi\right\|_{1,\varrho} \leqslant \int_0^\infty e^{\operatorname{Re}\lambda\theta} e^{\varrho\theta} \left|Q(\lambda)\varphi\right| d\theta = \frac{1}{\varrho + \operatorname{Re}\lambda} \left|Q(\lambda)\varphi\right|$$

shows that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\varrho$, this is the case if and only if $Q(\lambda)$ is a bounded linear operator from X into $\mathbb{R}^m$.

Proposition 3.8. *Let $k \in L^1_\varrho(\mathbb{R}_+; \mathbb{R}^{m\times m})$. Then $Q(\lambda)$ is a bounded linear operator from X into $\mathbb{R}^m$ for all λ with $\operatorname{Re}\lambda > -\varrho$.*

Proof. From the definition (3.25) of $Q(\lambda)$ we get

$$\begin{aligned}
\left|Q(\lambda)\varphi\right| &\leqslant \int_0^\infty e^{-\operatorname{Re}\lambda t} \int_{-\infty}^0 \left|k(t-\sigma)\right|\left|\varphi(\sigma)\right| d\sigma\, dt \\
&= \int_0^\infty e^{-(\operatorname{Re}\lambda+\varrho)t} \int_{-\infty}^0 e^{\varrho(t-\sigma)}\left|k(t-\sigma)\right| e^{\varrho\sigma}\left|\varphi(\sigma)\right| d\sigma\, dt \\
&= \int_{-\infty}^0 e^{\varrho\sigma}\left|\varphi(\sigma)\right| \int_{-\sigma}^\infty e^{-(\operatorname{Re}\lambda+\varrho)(\sigma+\tau)} e^{\varrho\tau}\left|k(\tau)\right| d\tau\, d\sigma \\
&\leqslant \|k\|_{1,\varrho}\|\varphi\|_{1,\varrho}
\end{aligned}$$

for all λ with $\operatorname{Re}\lambda > -\varrho$. $\square$

We collect our findings into a theorem.

Theorem 3.9. *Let $k \in L^1_{\varrho+\varepsilon}(\mathbb{R}_+; \mathbb{R}^{m\times m})$ for some $\varepsilon > 0$. A point $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\varrho$ belongs to the spectrum of A if and only if it is a root of the characteristic equation*

$$\det\bigl(I - \widehat{k}(\lambda)\bigr) = 0. \tag{3.30}$$

Every root of the characteristic equation in the right half-plane

$$\Pi_+^\varrho := \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > -\varrho\}$$

is an eigenvalue of A. There are at most finitely many roots of (3.30) in Π_+^ϱ.

The mapping $\lambda \mapsto R(\lambda, A)$ is meromorphic in Π_+^ϱ. More precisely, the mapping $\lambda \mapsto R(\lambda, A)$ is analytic in Π_+^ϱ except at the roots λ_ℓ of (3.30), where it has a pole of order at most equal to the multiplicity of λ_ℓ as a root of (3.30).

Proof. All statements except the last two have already been proven. To prove that there are only finitely many characteristic roots in Π_+^ϱ, notice that the analytic function $\det(I - \widehat{k}(\lambda))$ is not identically zero and therefore there are at most finitely many roots of (3.40) in any compact subset of $\Pi_+^{\varrho+\varepsilon}$. By the Riemann–Lebesgue lemma $\widehat{k}(\lambda)$ tends to zero as λ tends to infinity along a vertical line.

The convergence is uniform for lines in Π_+^ϱ. It follows that (3.40) has at most finitely many roots in the right half-plane Π_+^ϱ.

Finally, notice that the matrix valued function $\lambda \mapsto I - \widehat{k}(\lambda)$ is analytic in Π_+^ϱ and that it therefore follows from Cramer's rule that the inverse matrix $(I - \widehat{k}(\lambda))^{-1}$ can be expressed as an analytic matrix valued function divided by the scalar function $\det(I - \widehat{k}(\lambda))$. As a consequence, the statement is true for $(I - \widehat{k}(\lambda))^{-1}$ and the corresponding result for the resolvent $R(\lambda, A)$ follows from the representation (3.27). □

Because of lack of compactness we cannot draw any definite conclusions concerning the asymptotic behavior of the linearized semigroup from Theorem 3.9. Indeed, there are examples (in exponentially weighted L^1-spaces!, see [32, pp. 117–118]) of semigroups that do not decay exponentially even though the spectrum of the infinitesimal generator lies in the half-plane $\operatorname{Re}\lambda < -1$. Therefore we have to deduce the exponential decay of T in a different manner. We do it in two steps. We first show that the exponential decay of T follows from the exponential decay of the solution of the concrete linearized system (LRE)–(IC). Then we apply well-known theory of Volterra integral equations [19], most notably the Paley–Wiener theorem, to give conditions guaranteeing that the solutions of (LRE)–(IC) decay exponentially.

As the expression (3.27) shows, there is an intimate connection between deriving information by inverse Laplace transformation for the concrete equation (LRE) and for the corresponding semigroup. We refer to [2] for general abstract inverse Laplace transform techniques. Here we concentrate on (LRE).

Lemma 3.10. *Assume that the solution x of* (LRE)–(IC) *satisfies*

$$\int_0^\infty e^{\varrho\sigma} \left|x(\sigma)\right| d\sigma < \infty, \tag{3.31}$$

for all initial conditions φ. Then the solution semigroup T is uniformly exponentially stable, that is, there exist $\varepsilon > 0$ and $\widetilde{M} \geqslant 1$ such that

$$\left\|T(t)\right\|_{1,\varrho} \leqslant \widetilde{M} e^{-\varepsilon t}, \quad t > 0. \tag{3.32}$$

If, instead of (3.31), *the stronger condition*

$$\int_0^\infty e^{\varrho\sigma} \left|x(\sigma)\right| d\sigma \leqslant M\|\varphi\|_{1,\varrho} \tag{3.33}$$

holds for all φ, then (3.32) *holds with $\varepsilon = \varrho$.*

Proof. Recall that $T(t)\varphi = x_t$. Therefore

$$\begin{aligned} \left\|T(t)\varphi\right\|_{1,\varrho} &= \|x_t\|_{1,\varrho} = \int_{-\infty}^0 e^{\varrho\theta}\left|x(t+\theta)\right| d\theta = \int_{-\infty}^t e^{\varrho(\sigma-t)}\left|x(\sigma)\right| d\sigma \\ &= e^{-\varrho t}\int_{-\infty}^t e^{\varrho\sigma}\left|x(\sigma)\right| d\sigma = e^{-\varrho t}\left(\|\varphi\|_{1,\varrho} + \int_0^t e^{\varrho\sigma}\left|x(\sigma)\right| d\sigma\right). \end{aligned} \tag{3.34}$$

If (3.33) holds we are done. Under the weaker assumption (3.31), we still have

$$\int_0^\infty \|T(t)\varphi\|_{1,\varrho}\,dt < \infty$$

for all φ and so the Datko–Pazy theorem [32, Theorem 4.1, p. 116] implies that (3.32) holds. □

Let k be an $m \times m$-matrix valued function defined on $\mathbb{R}_+$. Its *resolvent kernel* is by definition the solution (which is unique if it exists [19, Lemma 3.3, p. 233]) of the equation

$$r = k + k * r = k + r * k, \tag{3.35}$$

where the star denotes convolution:

$$(f * g)(t) := \int_0^t f(t-s)g(s)\,ds. \tag{3.36}$$

The importance of the resolvent kernel stems from the fact that the solution of the linear Volterra equation

$$x = k * x + f \tag{3.37}$$

is given by

$$x = f + r * f. \tag{3.38}$$

This representation of the solution is particularly handy if $r \in L^1$ because then the solution x inherits the asymptotic behavior from the forcing function f. The Paley–Wiener theorem gives an easily applicable criterion for when the resolvent belongs to L^1. It says that if the kernel $k \in L^1$ and the equation

$$\det\bigl(I - \widehat{k}(\lambda)\bigr) = 0 \tag{3.39}$$

has no roots with $\operatorname{Re}\lambda \geqslant 0$, then $r \in L^1$. In practice the number of roots of (3.39) in the right half-plane can conveniently be determined by a variant of the argument principle called the Nyquist criterion [19, p. 61].

With this result in our arsenal we can easily prove the following result. It is a direct consequence of Theorem 4.11 in [19, p. 129] for general regular weight functions dominated by a certain submultiplicative weight function. But because our weight function is the exponential function, the proof is straightforward and we prefer to present it.

Lemma 3.11. *Let $k \in L^1_\varrho(\mathbb{R}_+; \mathbb{R}^{m\times m})$. If the characteristic equation*

$$\det\bigl(I - \widehat{k}(\lambda)\bigr) = 0 \tag{3.40}$$

has no roots with $\operatorname{Re}\lambda \geqslant -\varrho$, *then the solution x of* (LRE)–(IC) *satisfies the estimate*

$$\int_0^\infty e^{\varrho\sigma}\bigl|x(\sigma)\bigr|\,d\sigma \leqslant M\|\varphi\|_{1,\varrho} \tag{3.41}$$

for some $M \geqslant 1$.

Proof. Using (IC) we can rewrite (LRE) as

$$x(t) = \int_0^t k(s)x(t-s)\,ds + \int_{-\infty}^0 k(t-s)\varphi(s)\,ds. \tag{3.42}$$

Multiplying both sides of (3.42) by $e^{\varrho t}$ one finds that the function x^ϱ defined by

$$x^\varrho(t) = e^{\varrho t}x(t), \quad t \geqslant 0,$$

satisfies

$$x^\varrho(t) = \int_0^t k^\varrho(s)x^\varrho(t-s)\,ds + f^\varrho(t), \tag{3.43}$$

where

$$k^\varrho(t) = e^{\varrho t}k(t),$$
$$f^\varrho(t) = \int_{-\infty}^0 k^\varrho(t-s)e^{\varrho s}\varphi(s)\,ds.$$

Because $k \in L^1_\varrho$ we have $k^\varrho \in L^1$. We assume that (3.40) has no roots λ with $\operatorname{Re}\lambda \geqslant -\varrho$. Because

$$\widehat{k^\varrho}(\lambda) = \widehat{k}(\lambda - \varrho),$$

the equation

$$\det\bigl(I - \widehat{k^\varrho}(\lambda)\bigr) = 0$$

has no roots λ with $\operatorname{Re}\lambda \geqslant 0$. The Paley–Wiener theorem now yields that the resolvent kernel r^ϱ of k^ϱ belongs to L^1.

Multiplying both sides of (3.35) by $e^{\varrho t}$ one sees that

$$r^\varrho(t) = e^{\varrho t}r(t)$$

and, in particular,

$$\|r^\varrho\|_1 = \|r\|_{1,\varrho},$$

where r is the resolvent kernel of k. Applying the solution formula (3.38) to the Volterra equation (3.43) we obtain

$$\bigl|x^\varrho(t)\bigr| \leqslant \bigl|f^\varrho(t)\bigr| + \int_0^t \bigl|r^\varrho(t-s)\bigr|\bigl|f^\varrho(s)\bigr|\,ds. \tag{3.44}$$

It is plain to show that

$$\int_0^\infty \left| f^\varrho(t) \right| dt \leqslant \|k\|_{1,\varrho} \|\varphi\|_{1,\varrho} \tag{3.45}$$

and that, as a consequence,

$$\int_0^\infty \int_0^t \left| r^\varrho(t-s) \right| \left| f^\varrho(s) \right| ds\, dt \leqslant \|r\|_{1,\varrho} \|k\|_{1,\varrho} \|\varphi\|_{1,\varrho}. \tag{3.46}$$

Now (3.44), (3.45) and (3.46) imply (3.41). □

Theorem 3.12. *Let $k \in L^1_\varrho(\mathbb{R}_+; \mathbb{R}^{m\times m})$. If the characteristic equation*

$$\det\bigl(I - \widehat{k}(\lambda)\bigr) = 0 \tag{3.47}$$

has no roots with $\operatorname{Re} \lambda \geqslant -\varrho$, *then the solution semigroup T of the linearized problem satisfies*

$$\left\| T(t) \right\| \leqslant M e^{-\varrho t}, \quad t \geqslant 0, \tag{3.48}$$

for some $M \geqslant 1$.

Proof. This follows immediately from Lemmas 3.10 and 3.11. □

We now obtain the direct sum decomposition

$$X = X_+ \oplus X_- \tag{3.49}$$

into two T-invariant subspaces X_+ and X_- using abstract functional analytic results found for instance in [35] and [27]. Indeed, it follows from [35, Section VIII.8] and [27, Sections III.§6.4-5] that the operator P_+ defined by

$$P_+ = \frac{1}{2\pi i} \int_{\Gamma_+} R(\lambda, A)\, d\lambda, \tag{3.50}$$

where Γ_+ is a counter clock-wise contour in the open right half-plane $\Pi^\varrho_+ = \{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda > -\varrho\}$ surrounding all the eigenvalues of A in Π^ϱ_+, is a projection onto a closed subspace X_+ and that T leaves X_+ and X_- invariant. Here X_+ is the direct sum of the generalized eigenspaces corresponding to eigenvalues of A in Π^ϱ_+ and X_- is the complementary subspace defined by

$$X_- := P_- X$$

with

$$P_- := I - P_+.$$

Notice that substituting the expression (3.27) for the resolvent $R(\lambda, A)$ into (3.50) one obtains

$$(P_+\varphi)(\theta) = \frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda\theta}\big(1-\widehat{k}(\lambda)\big)^{-1} Q(\lambda)\varphi\,d\lambda, \quad \theta \leqslant 0 \tag{3.51}$$

(the last term on the right-hand side of (3.27) is holomorphic and therefore its contribution vanishes when integrated over Γ_+).

Recall from Theorem 3.9 that there are at most finitely many eigenvalues of A in Π_+^{ϱ} and that each of them is a pole of finite order of the integrand in (3.51). An application of the residue theorem therefore shows that there are vectors $\alpha_{\ell j}(\varphi)$ depending on φ, such that

$$\frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda t}\big(I-\widehat{k}(\lambda)\big)^{-1} Q(\lambda)\varphi\,d\lambda = \sum_{\ell=1}^{n}\sum_{j=0}^{p_\ell-1} \alpha_{\ell j}(\varphi)t^j e^{\lambda_\ell t} \tag{3.52}$$

for all $t \in \mathbb{R}$. Here p_ℓ is an integer not larger than the multiplicity of λ_ℓ as a root of (3.47). In particular,

$$(P_+\varphi)(\theta) = \sum_{\ell=1}^{n}\sum_{j=0}^{p_\ell-1} \alpha_{\ell j}(\varphi)\theta^j e^{\lambda_\ell\theta}, \quad \theta \leqslant 0, \tag{3.53}$$

and hence $X_+ = P_+X$ is finite dimensional.

Note that because P_+ is a projection, (3.53) implies that

$$\alpha_{\ell j}(\varphi) = \alpha_{\ell j}(P_+\varphi) \tag{3.54}$$

for all $\varphi \in X$.

Lemma 3.13. *Let $k \in L^1_\varrho(\mathbb{R}_+; \mathbb{R}^{m\times m})$. Then the restriction of the solution semigroup to X_- is uniformly exponentially stable, that is, there exist $M \geqslant 1$ and $\varepsilon > 0$ such that for all $\varphi_- \in X_-$*

$$\big\|T(t)\varphi_-\big\|_{1,\varrho} \leqslant Me^{-\varepsilon t}\|\varphi_-\|_{1,\varrho}, \quad t \geqslant 0. \tag{3.55}$$

Proof. By the obvious variant of Lemma 3.10, the result follows once we have shown that the solution x^- of (LRE) with initial condition

$$x^-(\theta) = P_-\varphi(\theta), \quad \theta \leqslant 0, \tag{3.56}$$

is an element of $L^1_\varrho(\mathbb{R}_+; \mathbb{R}^m)$ for all $\varphi \in X$. To prove this statement, first notice that because of the linearity of (LRE), the solution x of (LRE)–(IC) has the unique decomposition

$$x(t) = x^+(t) + x^-(t), \quad t \in \mathbb{R}, \tag{3.57}$$

where x^+ is the solution of (LRE) with initial condition

$$x^+(\theta) = P_+\varphi(\theta), \quad \theta \leqslant 0. \tag{3.58}$$

Theorem 2.5 of [19, p. 197] states that the solution $x(t)$ of the Volterra equation (3.23) (which, as we have shown, is equivalent to (LRE)–(IC)) has the form

$$x(t) = \sum_{\ell=1}^{n} \sum_{j=0}^{p_\ell - 1} \alpha_{\ell j}(\varphi) t^j e^{\lambda_\ell t} + \xi(\varphi; t), \quad t > 0, \tag{3.59}$$

where $\xi(\varphi, \cdot) \in L^1_\varrho(\mathbb{R}_+; \mathbb{R}^m)$ and the coefficients $\alpha_{\ell j}(\varphi)$ have been obtained by exactly the same application of the residue theorem as above (so our notation is consistent). In particular,

$$x^+(t) = \sum_{\ell=1}^{n} \sum_{j=0}^{p_\ell - 1} \alpha_{\ell j}(\varphi) t^j e^{\lambda_\ell t} + \xi(P_+\varphi; t), \quad t > 0, \tag{3.60}$$

where we have used (3.54). Combining (3.57), (3.59) and (3.60), we conclude that

$$x^-(t) = x(t) - x^+(t) = \xi(\varphi; t) - \xi(P_+\varphi; t), \quad t > 0,$$

and hence that x^- is an element of $L^1_\varrho(\mathbb{R}_+; \mathbb{R}^m)$. This completes the proof. $\square$

Remark 3.14. As a matter of fact, one can prove that $\xi(P_+\varphi; t) = 0$ and hence that

$$x^+(t) = \sum_{\ell=1}^{n} \sum_{j=0}^{p_\ell - 1} \alpha_{\ell j}(\varphi) t^j e^{\lambda_\ell t}, \quad t > 0,$$

but this fact does not play any role in the proof of the principle of linearized stability.

Note that a more detailed characterization of the $\alpha_{\ell j}$ can be given, as elaborated in detail for delay-differential equations in [28], see also [16].

We are now ready to prove the *principle of linearized stability*.

Theorem 3.15. *Let $\varrho > 0$ and assume that $F : L^1_\varrho(\mathbb{R}_-; \mathbb{R}^m) \to \mathbb{R}^m$ is continuously Fréchet differentiable. Let $\bar{x}$ be a steady-state of* (RE). *Let $k \in L^\infty_\varrho(\mathbb{R}_+; \mathbb{R}^{m\times m})$ represent $DF(\bar{x})$:*

$$DF(\bar{x})\varphi = \langle \varphi, k \rangle = \int_0^\infty k(s)\varphi(-s)\, ds.$$

(a) *If all the roots of the characteristic equation* (3.40) *have negative real part, then the steady-state $\bar{x}$ is exponentially stable.*
(b) *If there exists at least one root of* (3.40) *with positive real part, then the steady-state $\bar{x}$ is unstable.*

Proof. (a) In Theorem 3.9 we showed that (3.40) has at most finitely many roots in any right half-plane. If there are no roots with nonnegative real part we can therefore choose $\bar{\varrho} \in (0, \varrho)$ such that (3.40) has no roots λ with $\operatorname{Re}\lambda \geqslant -\bar{\varrho}$.

The constant function $\bar{x}$ obviously belongs to $L^1_{\bar{\varrho}}$. Corollary 3.2 shows that F, considered as a function on $L^1_{\bar{\varrho}}$, is Fréchet differentiable at $\bar{x}$.

Now Theorem 3.12 yields that the semigroup of the linearized problem satisfies

$$\|T(t)\| \leqslant M e^{-\bar{\varrho} t}, \quad t \geqslant 0,$$

for some $M \geqslant 1$. It follows that the conditions (i) and (ii) of Theorem 2.4 are satisfied with X_+ equal to the trivial subspace $\{0\}$. Hence $\bar{x}$ is exponentially stable because $\sigma(A|_{X_+})$ is empty. This proves (a).

(b) We assume that the characteristic equation (3.40) has n roots $\lambda_1, \ldots, \lambda_n$ in the right half-plane $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > -\varrho\}$ and consider the decomposition (3.49). As we have already noted it follows from the general theory that X_+ is finite dimensional. By Lemma 3.13, $T_- := T|_{X_-}$ is uniformly exponentially stable. Thus the decomposition (3.49) satisfies the conditions (i) and (ii) of Theorem 2.4. If there exists at least one characteristic root with positive real part, then this root is an eigenvalue of $A|_{X_+}$ and (b) follows from Theorem 2.4(b). $\square$

4. Delay-differential equations

In this section we consider the delay-differential equation

$$\dot{y}(t) = F(y_t), \quad t > 0, \tag{DDE}$$

with infinite delay and initial condition

$$y(\theta) = \psi(\theta), \quad -\infty < \theta \leqslant 0. \tag{IC}$$

The case of finite delay was treated in the book [13] by adjoint semigroup methods. In this section we show that on the abstract side the only essential difference between the treatment of delay-differential equations and renewal equations is the choice of the spaces X and X' and the unperturbed semigroup. On the concrete side we use the *differential resolvent kernel* instead of the usual resolvent kernel to express the solution of the linearization of (DDE) and Grossman and Miller's [18] criterion for integrability of the differential resolvent kernel, which is analogous to the Paley–Wiener theorem for integrability of the ordinary resolvent kernel.

We like to point out that Hino et al. in Section 4.2 of [25] derive, following Murakami [30], a variation-of-constants formula in the second dual space (see also [26]).

Analogously to the treatment of the renewal equation in Section 3, we show that the problem (DDE)–(IC) is equivalent to the abstract integral equation

$$u(t) = T_0(t)\psi + j^{-1}\int_0^t T_0'^{*}(t-s)(\ell \circ F)\big(u(s)\big)\,ds, \tag{AIE}$$

with a particular choice of the spaces X and X', the unperturbed semigroup T_0 and the linear injection ℓ.

Let $\varrho > 0$. As the space X we choose the Banach space $C_{0,\varrho}(\mathbb{R}_-;\mathbb{R}^m)$ of all $\mathbb{R}^m$-valued functions ψ defined on $\mathbb{R}_-$ such that $\theta \mapsto e^{\varrho\theta}\psi(\theta)$ is continuous and vanishes at minus infinity with norm

$$\|\psi\|_{\infty,\varrho} = \sup_{\theta\in\mathbb{R}_-} e^{\varrho\theta}\big|\psi(\theta)\big|.$$

The normed dual X^* of $C_{0,\varrho}(\mathbb{R}_-;\mathbb{R}^m)$ is represented by the space $M_\varrho(\mathbb{R}_+;\mathbb{R}^{m\times m})$ of all $m \times m$ matrices with entries that are measures μ_{ij} on $\mathbb{R}_+$ such that

$$\|\mu\|_\varrho := \int_{\mathbb{R}_+} e^{\varrho\theta}\big|\mu(d\theta)\big| < \infty.$$

The pairing between X and X^* is given by

$$\langle \psi, \mu \rangle = \int_{\mathbb{R}_+} \mu(d\theta)\, \psi(-\theta)$$

and the norm of $\mu \in M_\varrho(\mathbb{R}_+; \mathbb{R}^{m\times m})$ by $\|\mu\|_\varrho$. The linearized version of (DDE) therefore reads

$$\dot{y}(t) = \int_{\mathbb{R}_+} \mu(d\theta)\, y(t-\theta), \tag{LDDE}$$

where μ is the representation of $DF(\overline{\psi})$ for a steady-state $\overline{\psi}$.

The unperturbed semigroup is again a family of shift operators, but unlike the semigroup (3.9) the extension is now by the value at zero:

$$\big(T_0(t)\psi\big)(\theta) = \begin{cases} \psi(t+\theta), & -\infty < \theta \leqslant -t, \\ \psi(0), & -t < \theta \leqslant 0, \end{cases} \qquad \psi \in X,\ t \geqslant 0. \tag{4.1}$$

As the space X' we choose $X' = \mathbb{R}^m \times L^1_\varrho(\mathbb{R}_+; \mathbb{R}^m)$ with pairing

$$\big\langle \psi, (c, g)\big\rangle = \psi(0)\cdot c + \int_0^\infty \psi(-\theta)\cdot g(\theta)\, d\theta, \quad \psi \in X,\ (c, g) \in X', \tag{4.2}$$

and norm

$$\big\|(c, g)\big\| = |c| + \|g\|_{1,\varrho},$$

and as X'^* we take $\mathbb{R}^m \times L^\infty_\varrho(\mathbb{R}_-; \mathbb{R}^m)$ with pairing

$$\big\langle (c, g), (d, f)\big\rangle = c\cdot d + \int_0^\infty g(\theta)\cdot f(-\theta)\, d\theta.$$

With respect to the pairing (4.2) the adjoint semigroup T_0' of T_0 is given by

$$T_0'(t)(c, g) = \left(c + \int_0^t g(s)\, ds, g_t\right), \quad (c, g) \in X',\ t \geqslant 0.$$

It is strongly continuous.

Finally we define the injection $\ell : \mathbb{R}^m \to X'^*$ by

$$\ell(c) = (c, 0), \quad c \in \mathbb{R}^m. \tag{4.3}$$

We can now easily prove the analogues of Lemmas 3.3 and 3.4, Corollaries 3.5 and 3.6 and Theorem 3.7.

Lemma 4.1. *For every $z \in \mathbb{R}^m$ and $(c, g) \in X'$ one has*

$$\langle T_0'^*(t)\ell z, (c, g)\rangle = z \cdot \left(c + \int_0^t g(s)\,ds\right), \quad t \geqslant 0.$$

Proof.

$$\begin{aligned}\langle T_0'^*(t)\ell z, (c, g)\rangle &= \langle \ell z, T_0'(t)(c, g)\rangle \\ &= \left\langle (z, 0), \left(c + \int_0^t g(s)\,ds, g_t\right)\right\rangle = z \cdot \left(c + \int_0^t g(s)\,ds\right). \qquad \square\end{aligned}$$

Lemma 4.2. *Let $h : \mathbb{R}_+ \to \mathbb{R}^m$ be a continuous function. Then, for every $(c, g) \in X'$ one has*

$$\left\langle \int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau, (c, g)\right\rangle = c \cdot \int_0^t h(\tau)\,d\tau + \int_0^t \int_0^{t-s} h(\sigma)\,d\sigma\, g(s)\,ds, \quad t \geqslant 0.$$

Proof. From Lemma 4.1 it follows immediately that

$$\begin{aligned}\left\langle \int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau, (c, g)\right\rangle &= \int_0^t \langle T_0'^*(t-\tau)\ell h(\tau), (c, g)\rangle\,d\tau \\ &= \int_0^t h(\tau) \cdot \left(c + \int_0^{t-\tau} g(s)\,ds\right) d\tau \\ &= c \cdot \int_0^t h(\tau)\,d\tau + \int_0^t \int_0^{t-s} h(\sigma)\,d\sigma \cdot g(s)\,ds. \qquad \square\end{aligned}$$

Corollary 4.3. *Let $h : \mathbb{R}_+ \to \mathbb{R}^m$ be a continuous function and define $\psi \in X$ by*

$$\psi(\theta) = \begin{cases} \int_0^{t+\theta} h(\sigma)\,d\sigma, & -t \leqslant \theta \leqslant 0, \\ 0, & -\infty < \theta < -t. \end{cases} \tag{4.4}$$

Then

$$\int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau = j\psi. \tag{4.5}$$

In particular, $\int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau \in j(X)$ and

$$\left\| j^{-1}\left(\int_0^t T_0'^*(t-\tau)\ell h(\tau)\,d\tau\right)\right\|_{\infty,\varrho} \leqslant \int_0^t |h(s)|\,ds, \quad t \geqslant 0, \tag{4.6}$$

that is, Hypothesis 2.1 *holds with $M = 1$ and $\omega = 0$.*

Proof. For every $(c, g) \in X'$ we have by (4.4) and Lemma 4.2:

$$\begin{aligned}\langle \psi, (c, g) \rangle &= \psi(0) \cdot c + \int_0^\infty \psi(-\theta) \cdot g(\theta)\, d\theta \\ &= c \cdot \int_0^t h(\sigma)\, d\sigma + \int_0^t \int_0^{t-\theta} h(\sigma)\, d\sigma \cdot g(\theta)\, d\theta \\ &= \left\langle \int_0^t T_0'^*(t-\tau)\ell h(\tau)\, d\tau, (c, g) \right\rangle,\end{aligned}$$

from which (4.5) follows. The estimate (4.6) follows immediately:

$$\begin{aligned}\left\| j^{-1}\left(\int_0^t T_0'^*(t-\tau)\ell h(\tau)\, d\tau \right) \right\|_{\infty,\varrho} &= \|\psi\|_{\infty,\varrho} = \sup_{-t \leqslant \theta \leqslant 0} e^{\varrho\theta} \left| \int_0^{t+\theta} h(\sigma)\, d\sigma \right| \\ &\leqslant \sup_{-t \leqslant \theta \leqslant 0} e^{\varrho\theta} \int_0^{t+\theta} \left| h(\sigma) \right| d\sigma = \int_0^t \left| h(\sigma) \right| d\sigma. \qquad \square\end{aligned}$$

Corollary 4.4. *Let $u : \mathbb{R}_+ \to X$ be continuous. Then*

$$\left(j^{-1} \int_0^t T_0'^*(t-s)\ell F\big(u(s)\big)\, ds \right)(\theta) = \begin{cases} \int_0^{t+\theta} F(u(s))\, ds, & -t \leqslant \theta \leqslant 0, \\ 0, & -\infty < \theta < -t. \end{cases}$$

Theorem 4.5. *Let $\psi \in X = C_{0,\varrho}(\mathbb{R}_-; \mathbb{R}^m)$ be given.*

(a) *Suppose that $y : \mathbb{R} \to \mathbb{R}^m$ is continuous and satisfies the integrated version of* (DDE)–(IC)*. Then the function $u : [0, \infty) \to X$ defined by $u(t) := y_t$ is continuous and satisfies* (AIE)*.*
(b) *If $u : [0, \infty) \to X$ is continuous and satisfies* (AIE)*, then the function $y : \mathbb{R} \to \mathbb{R}^m$ defined by*

$$y(t) := \begin{cases} \psi(t) & \text{for } -\infty < t < 0, \\ u(t)(0) & \text{for } t \geqslant 0 \end{cases} \tag{4.7}$$

is continuous and satisfies (DDE)–(IC)*.*

Proof. With Corollary 4.4 at hand the proof is completely analogous to the proof of Theorem 3.7 (see also [13, Theorem 4.1]). $\square$

The characterization of the part of the point spectrum of the infinitesimal generator A of the linearized semigroup T in $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant -\varrho\}$ and the growth bound of T follows the lines of the corresponding analysis for renewal equations in Section 3. Exactly as in (3.22) we get

$$\big(R(\lambda, A)\psi\big)(\theta) = e^{\lambda\theta}\left(\widehat{y}(\lambda) + \int_\theta^0 e^{-\lambda\sigma}\psi(\sigma)\, d\sigma \right). \tag{4.8}$$

By (4.7) the linearized equation (LDDE) can be rewritten as

$$\dot{y}(t) = \int\limits_{[0,t)} \mu(ds)\, y(t-s) + \int\limits_{[t,\infty)} \mu(ds)\, \psi(t-s). \tag{4.9}$$

Taking the Laplace transform of both sides of (4.9), solving for $\widehat{y}(\lambda)$ and substituting the obtained expression for $\widehat{y}(\lambda)$ into (4.8) one obtains the analogue of (3.27), viz.

$$\big(R(\lambda, A)\psi\big)(\theta) = e^{\lambda\theta}\big(\lambda I - \widehat{\mu}(\lambda)\big)^{-1} Q(\lambda)\psi + \big(H(\lambda)\psi\big)(\theta), \tag{4.10}$$

where

$$Q(\lambda)\psi = \psi(0) + \int\limits_0^\infty e^{-\lambda t} \int\limits_{[t,\infty)} \mu(ds)\, \psi(t-s)\, dt,$$

$$\big(H(\lambda)\psi\big)(\theta) = e^{\lambda\theta} \int\limits_\theta^0 e^{-\lambda\sigma}\, \psi(\sigma)\, d\sigma.$$

Exactly as in Section 3 one shows that $Q(\lambda): X \to \mathbb{R}^m$ and $H(\lambda): X \to X$ are bounded linear operators for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\varrho$. It follows as in Section 3 that in this half-plane, the resolvent $R(\lambda, A)$ exists as bounded linear operator from X into X if and only if λ is not a root of the characteristic equation

$$\det\big(\lambda I - \widehat{\mu}(\lambda)\big) = 0. \tag{4.11}$$

The intersection of the spectrum of A with the right half-plane $\Pi_+^\varrho = \{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda > -\varrho\}$ therefore consists of the roots of (4.11). They are isolated points of the spectrum and in fact eigenvalues of A.

The solution of the linear Volterra integro-differential equation

$$\dot{y}(t) = \int\limits_{[0,t)} \mu(ds)\, y(t-s) + f(t), \quad t > 0, \qquad y(0) = y_0 \tag{4.12}$$

can be expressed by the formula

$$y(t) = r(t)y_0 + \int\limits_0^t r(t-s) f(s)\, ds, \quad t \geqslant 0, \tag{4.13}$$

where the $m \times m$-matrix valued function r is the *differential resolvent kernel* or *fundamental solution* of μ, that is, the solution of

$$\dot{r}(t) = \int\limits_{[0,t)} \mu(ds)\, r(t-s), \quad t > 0, \qquad r(0) = I \tag{4.14}$$

(see [19]). The Grossman–Miller theorem (see [18] and [19, Theorem 3.5, p. 83]) says that if $\mu \in M(\mathbb{R}_+; \mathbb{R}^{m\times m})$ then $r \in L^1(\mathbb{R}_+; \mathbb{R}^{m\times m})$ if and only if the characteristic equation (4.11) has no roots with nonnegative real part. When $r \in L^1(\mathbb{R}_+; \mathbb{R}^{m\times m})$, then also $\dot{r} \in L^1(\mathbb{R}_+; \mathbb{R}^{m\times m})$, so, in particular

$r \in L^\infty(\mathbb{R}_+; \mathbb{R}^{m\times m})$. Using this result instead of the Paley–Wiener theorem we can now prove the following analogue of Theorem 3.12.

Theorem 4.6. *Let $\mu \in M_\varrho(\mathbb{R}_+; \mathbb{R}^{m\times m})$. If the characteristic equation*

$$\det\bigl(\lambda I - \widehat{\mu}(\lambda)\bigr) = 0$$

has no roots with $\operatorname{Re}\lambda \geqslant -\varrho$, then the solution semigroup T of the linearized problem satisfies

$$\bigl\|T(t)\psi\bigr\| \leqslant Me^{-\varrho t}, \quad t \geqslant 0, \tag{4.15}$$

for some $M \geqslant 1$.

Proof. Let y be the solution of (4.9). First notice that

$$\begin{aligned}\bigl\|T(t)\psi\bigr\|_{\infty,\varrho} = \|y_t\|_{\infty,\varrho} &= \sup_{\theta\leqslant 0} e^{\varrho\theta}\bigl|y(t+\theta)\bigr| = e^{-\varrho t}\sup_{\theta\leqslant 0} e^{\varrho(t+\theta)}\bigl|y(t+\theta)\bigr| \\ &= e^{-\varrho t}\max\Bigl\{\sup_{\theta\leqslant 0}\bigl|\psi(\theta)\bigr|, \sup_{0<s\leqslant t} e^{\varrho s}\bigl|y(s)\bigr|\Bigr\} \\ &= e^{-\varrho t}\max\Bigl\{\|\psi\|_{\infty,\varrho}, \sup_{0<s\leqslant t} e^{\varrho s}\bigl|y(s)\bigr|\Bigr\}.\end{aligned}$$

To prove (4.15) we therefore have to show that

$$\sup_{s\in\mathbb{R}_+} e^{\varrho s}\bigl|y(s)\bigr| \leqslant M\|\psi\|_{\infty,\varrho} \tag{4.16}$$

for some M.

Let r be the differential resolvent kernel of μ. Multiplying the formula (4.13) by $e^{\varrho t}$ we obtain

$$e^{\varrho t}y(t) = r^\varrho(t)\psi(0) + \int_0^t r^\varrho(t-s)f^\varrho(s)\,ds, \tag{4.17}$$

where

$$\begin{aligned}r^\varrho(t) &= e^{\varrho t}r(t), \\ f^\varrho(t) &= e^{\varrho t}\int_{[t,\infty)} \mu(d\theta)\,\psi(s-\theta).\end{aligned}$$

We now claim that $r^\varrho \in L^1(\mathbb{R}_+; \mathbb{R}^{m\times m})$. Clearly $r^\varrho(0) = I$ and

$$\dot r^\varrho(t) = \varrho r^\varrho(t) + e^{\varrho t}\dot r(t) = \varrho r^\varrho(t) + \int_{[0,t)} e^{\varrho s}\mu(ds)\,r^\varrho(t-s).$$

Therefore r^ϱ is the differential resolvent kernel of the measure $\mu^\varrho(ds) := \varrho I\delta(ds) + e^{\varrho s}\mu(ds)$. The Laplace transform of μ^ϱ is given by

$$\widehat{\mu^\varrho}(\lambda) = \varrho I + \widehat{\mu}(\lambda - \varrho)$$

and the corresponding characteristic equation is

$$\det\bigl((\lambda-\varrho)I-\widehat{\mu}(\lambda-\varrho)\bigr)=0. \tag{4.18}$$

Because we assume that (4.11) has no roots with $\operatorname{Re}\lambda \geqslant -\varrho$, Eq. (4.18) has no roots with $\operatorname{Re}\lambda \geqslant 0$. By the Grossman–Miller theorem $r^{\varrho} \in L^1(\mathbb{R}_+;\mathbb{R}^{m\times m}) \cap L^{\infty}(\mathbb{R}_+;\mathbb{R}^{m\times m})$.

Because $r^{\varrho} \in L^{\infty}(\mathbb{R}_+;\mathbb{R}^{m\times m})$ the first term on the right-hand side of (4.17) is bounded by $M\|\psi\|_{\infty,\varrho}$ for some M. Because $r^{\varrho} \in L^1(\mathbb{R}_+;\mathbb{R}^{m\times m})$ the same is true for the second term if $f^{\varrho}(s)$ is bounded by $M\|\psi\|_{\infty,\varrho}$ for some M. But

$$\bigl|f^{\varrho}(s)\bigr| \leqslant \int\limits_{[s,\infty)} e^{\theta s}|\mu|(d\theta)\,e^{\varrho(s-\theta)}|\psi|(s-\theta) \leqslant \|\mu\|_{\varrho}\|\psi\|_{\infty,\varrho}.$$

This completes the proof. □

With the preceding results at hand, the principle of linearized stability can be proved in a way completely analogous to the proof of Theorem 3.15. We just state the result.

Theorem 4.7. *Let $\varrho > 0$ and assume that $F : C_{0,\varrho}(\mathbb{R}_-;\mathbb{R}^m) \to \mathbb{R}^m$ is continuously Fréchet differentiable. Let $\bar{x}$ be a steady-state of* (DDE). *Let $\mu \in M_{\varrho}(\mathbb{R}_+;\mathbb{R}^{m\times m})$ represent $DF(\bar{x})$:*

$$DF(\bar{x})\psi = \langle\psi,\mu\rangle = \int\limits_0^{\infty} \mu(ds)\,\psi(-s).$$

(a) *If all the roots of the characteristic equation* (4.11) *have negative real part, then the steady-state $\bar{x}$ is exponentially stable.*
(b) *If there exists at least one root of* (4.11) *with positive real part, then the steady-state $\bar{x}$ is unstable.*

5. Coupled renewal and delay-differential equations

In Sections 3 and 4 we treated the cases of nonlinear renewal and delay-differential equations separately. In many applications, notably in population dynamics, the models appear as systems of coupled renewal and delay-differential equations [14,15,20,21]. It is straightforward to combine the results of Sections 3 and 4 to obtain results for such "mixed" equations. In this short section we therefore only present the results without proof.

Alternatively, we could have formulated all our results in terms of 'trivial' semigroups perturbed, at the level of the generator, by operators with *finite dimensional range* which are bounded as maps from X into X'^{*}, cf. [13]. It is the finite dimensionality of the range that allows one to obtain a characteristic equation.

We consider the system

$$x(t) = F_1(x_t, y_t), \tag{5.1}$$

$$\dot{y}(t) = F_2(x_t, y_t) \tag{5.2}$$

with initial condition

$$x(\theta) = \varphi(\theta), \quad -\infty < \theta \leqslant 0, \tag{5.3}$$

$$y(\theta) = \psi(\theta), \quad -\infty < \theta \leqslant 0. \tag{5.4}$$

As state space we choose $X = X_1 \times X_2$ with $X_1 = L^1_\varrho(\mathbb{R}_-;\mathbb{R}^m)$ and $X_2 = C_{0,\varrho}(\mathbb{R}_-;\mathbb{R}^n)$. The function F_1 maps X into $\mathbb{R}^m$ and F_2 maps X into $\mathbb{R}^n$. The map

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} : X \to \mathbb{R}^m \times \mathbb{R}^n$$

is defined in the obvious manner.

Let T_{10} be the C_0-semigroup defined on X_1 by (3.9) and let T_{20} be the C_0-semigroup defined on X_2 by (4.1). The two semigroups T_{10} and T_{20} induce in an obvious way a semigroup T_0 on X:

$$T_0(t) = \begin{pmatrix} T_{10}(t) & 0 \\ 0 & T_{20}(t) \end{pmatrix}. \tag{5.5}$$

The natural choice of X' is of course $X'_1 \times X'_2$ and X'^* is (isometrically isomorphic to) $X'^*_1 \times X'^*_2$. The embeddings $j_1 : X_1 \to X'^*_1$ and $j_2 : X_2 \to X'^*_2$ induce an embedding $j : X \to X'^*$ in the obvious way. Finally, let $\ell_1 : \mathbb{R}^m \to X'^*_1$ and $\ell_2 : \mathbb{R}^n \to X'^*_2$ be the injections defined by (3.13) and (4.3), respectively, and define

$$\ell \begin{pmatrix} z \\ c \end{pmatrix} = \begin{pmatrix} \ell_1 z \\ \ell_2 c \end{pmatrix}.$$

The system (5.1)–(5.4) is equivalent to the abstract integral equation

$$u(t) = T_0(t)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} + j^{-1}\left(\int_0^t T'^*_0(t-s)\ell \circ F\big(u(s)\big)\, ds \right). \tag{AIE}$$

If F is Fréchet differentiable, its derivative can be represented by the $(m+n) \times (m+n)$ matrix

$$\begin{pmatrix} k_{11} & \mu_{12} \\ k_{21} & \mu_{22} \end{pmatrix}, \tag{5.6}$$

where k_{11} and k_{21} are $m \times m$, resp., $m \times n$ matrices of elements of $L^\infty_\varrho(\mathbb{R}_+)$ and μ_{12} and μ_{22} are $n \times m$, resp., $m \times m$ matrices of elements in $M_\varrho(\mathbb{R}_+)$. The interpretation of (5.6) is that

$$DF(\overline{\varphi}, \overline{\psi})\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \int_0^\infty k_{11}(\theta)\varphi(-\theta)\, d\theta + \int_{\mathbb{R}_+} \mu_{12}(d\theta)\, \psi(-\theta) \\ \int_0^\infty k_{21}(\theta)\varphi(-\theta)\, d\theta + \int_{\mathbb{R}_+} \mu_{22}(d\theta)\, \psi(-\theta) \end{pmatrix}. \tag{5.7}$$

Linearizing the concrete system (5.1)–(5.2) about a steady-state one obtains

$$x(t) = \int_0^\infty k_{11}(\theta)x(t-\theta)\, d\theta + \int_{\mathbb{R}_+} \mu_{12}(d\theta)\, y(t-\theta), \tag{5.8}$$

$$\dot{y}(t) = \int_0^\infty k_{21}(\theta)x(t-\theta)\, d\theta + \int_{\mathbb{R}_+} \mu_{22}(d\theta)\, y(t-\theta). \tag{5.9}$$

Let

$$M(\lambda) = \begin{pmatrix} I & 0 \\ 0 & \lambda I \end{pmatrix} - \begin{pmatrix} \widehat{k}_{11}(\lambda) & \widehat{\mu}_{12}(\lambda) \\ \widehat{k}_{21}(\lambda) & \widehat{\mu}_{22}(\lambda) \end{pmatrix}.$$

Exactly as in Sections 3 and 4 one gets the following expression for the resolvent of the infinitesimal generator of the linearized semigroup associated with the system (5.8)–(5.9):

$$\left(R(\lambda, A)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)(\theta) = e^{\lambda t}M(\lambda)^{-1}Q(\lambda)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + H(\lambda)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(\theta), \tag{5.10}$$

where now the operator $Q(\lambda): X \to \mathbb{R}^{m+n}$ is defined by

$$\left(Q(\lambda)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)(\theta)$$
$$= \begin{pmatrix} \int_0^\infty e^{-\lambda t}\int_{-\infty}^0 k_{11}(t-s)\varphi(s)\,ds\,dt & \int_0^\infty e^{-\lambda t}\int_{[t,\infty)}\mu_{12}(ds)\,\psi(t-s)\,dt \\ \int_0^\infty e^{-\lambda t}\int_{-\infty}^0 k_{21}(t-s)\varphi(s)\,ds\,dt & \psi(0) + \int_0^\infty e^{-\lambda t}\int_{[t,\infty)}\mu_{22}(ds)\,\psi(t-s)\,dt \end{pmatrix}$$

and

$$H(\lambda)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = e^{\lambda\theta}\int_\theta^0 e^{-\lambda\sigma}\begin{pmatrix}\varphi(\sigma)\\ \psi(\sigma)\end{pmatrix}d\sigma.$$

Exactly as before one shows that $Q(\lambda)$ and $H(\lambda)$ are well-defined bounded linear operators for $\operatorname{Re}\lambda > -\varrho$ and hence that in this half-plane the spectral values are eigenvalues and precisely the roots of the characteristic equation

$$\det M(\lambda) = 0. \tag{5.11}$$

The principle of linearized stability for (5.1)–(5.2) says that if all the roots of (5.11) have negative real part, then the steady-state is stable whereas it is unstable if there exists at least one root with positive real part. The proof is a straightforward adaptation of the proof of Theorem 3.15.

6. Concluding remarks

Even though the life time of an organism is never infinite, it is often attractive to formulate population models without building in an *a priori* bound on the age, simply to have less parameters and a lower dimensional individual state space. But of course one needs (at least) the usual local stability and bifurcation results, in particular the principle of linearized stability and the Hopf bifurcation theorem.

Originally we were quite convinced that we would find the principle of linearized stability for differential equations with infinite delay in the published literature. But our search in [19,24,25,33] and in the references given in these sources, revealed only stability results in terms of Lyapunov functionals and no instability result at all. The Hopf bifurcation theorem, on the other hand, is proved in [34] by way of Lyapunov–Schmidt reduction for rather general equations.

A convenient way to derive stability and bifurcation results for equations with finite delay, is to use the theory of adjoint semigroups [31]. The key step is to reformulate the problem as an abstract integral equation. For such integral equations one can then show that information about the linearized problem does indeed tell us how solutions of the nonlinear problem behave locally. This is elaborated in detail in [13].

In this paper we have shown that a variant of this approach still works if the delay is infinite. Now, however, it is not a matter of applying a general abstract result in the context of a particular example. We have to use some of the special features of delay equations, in particular that they are generated by finite dimensional perturbations of the generator of an almost trivial semigroup.

Since the concrete delay equations and the abstract integral equation are equivalent, we can switch between the two formulations as much as we want. Here we have taken advantage of this freedom by studying the linear problem in terms of (inverse) Laplace transforms, i.e., by using results from [19] instead of performing a spectral analysis with due attention to the essential spectrum. So in essence we use the abstract integral equation only to "lift" linear information to the nonlinear equation (which is nontrivial when it concerns instability).

A centre manifold theorem for equations with infinite delay can be formulated and proved along the lines of [13] or [29]. Next one can combine the Hopf bifurcation theorem for ODEs with centre manifold reduction to derive the Hopf bifurcation theorem for the delay equations. The advantage of this approach is that stability information can be obtained more easily than by way of Lyapunov–Schmidt reduction.

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