Department of Mathematics and Statistics
Semigroups and Delay Equations
Exercises 7
28.3.2014

1. Let T be the translation semigroup with extension by zero on $L^1[-h, 0]$ as in Exercise 1.2. Characterise the spaces X^*, $X^\odot$, $X^{\odot *}$, $X^{\odot\odot}$, the semigroups T^*, $T^\odot$, $T^{\odot *}$, $T^{\odot\odot}$, and the generators A^*, $A^\odot$, $A^{\odot *}$, $A^{\odot\odot}$ providing all the details.

2. Let $X = C_0(\mathbb{R})$ and let P_t be the Poisson kernel

$$P_t(y) = \frac{1}{\pi} \frac{t}{t^2 + y^2}.$$

Let $*$ denote convolution:

$$\begin{aligned}(f * g)(x) &= \int_{\mathbb{R}} f(x - y) g(y) dy, \\ (f * \mu)(x) &= \int_{\mathbb{R}} f(x - y) d\mu(y),\end{aligned}$$

whenever the integrals exist for functions f and g and measureμ. Define $T(0) = I$ and

$$T(t)f = P_t * f, \quad f \in X, \quad t > 0.$$

Prove that

(a) T is a strongly continuous semigroup on X,
(b) $T^*(t)\mu = P_t * \mu$,
(c) $X^\odot = L^1(\mathbb{R})$,
(d) $T^*(t)X^* \subset X^\odot$ for all $t > 0$.

3. Let c_0 be the Banach space of all complex sequences converging to zero equipped with the supremum norm and let ℓ^p be the Banach space of all p-summable sequences with the norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \qquad 1 \le p < \infty.$$

Let X be either c_0 or ℓ^p $(1 \le p < \infty)$. Define for $x = \{x_1, x_2, \ldots\} \in X$

$$\left(T(t)x\right)_n = e^{-nt} x_n$$

Show that

(a) T is a strongly continuous semigroup on X,
(b) $c_0^\odot = \ell^1$,
(c) $(\ell^1)^\odot = c_0$,
(d) $(\ell^p)^\odot = \ell^q$ for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.