Department of Mathematics and Statistics
Semigroups and Delay Equations
Exercises 6
14.3.2014

1. Let A be a closed densely defined operator in the Banach space X and let Y be a Banach space that is continuously embedded in X. The *part* $A_|$ of A in Y is, by definition, the operator

$$\begin{aligned} A_| \varphi &= A\varphi \\ \mathcal{D}(A_|) &= \{\varphi \in \mathcal{D}(A) \cap Y : A\varphi \in Y\}. \end{aligned}$$

Show that if $\lambda \in \varrho(A)$ is such that $R(\lambda, A)Y \subset Y$, then $\lambda \in \varrho(A_|)$ and $R(\lambda, A_|) = R(\lambda, A)_|$.

2. Prove that $A^\odot$ is the part of A^* in $X^\odot$ and use this and the previous excercise to show that $\sigma(A^\odot) = \sigma(A^*)$.

3. Let A be the generator of the semigroup T defined by

$$T(t)\varphi = T_0(t)\varphi + j^{-1}\left(\int_0^t T_0^{\odot *} t - s)BT(s)\varphi ds\right).$$

and let B be given by

$$B\varphi = \sum_{i=1}^{N} \langle \varphi, r_i^* \rangle r_i^{\odot *}, \quad \varphi \in X.$$

Define

$$\begin{aligned} r_i(\lambda) &= j^{-1}R(\lambda, A_0^{\odot *})r_i^{\odot *}, \quad \lambda \in \varrho(A_0), \\ r_i^\odot(\lambda) &= R(\lambda, A_0^*)r_i^*, \quad \lambda \in \varrho(A_0) \end{aligned}$$

and let $M(\lambda)$ be the matrix with entries

$$M_{ij}(\lambda) = \langle r_j(\lambda), r_i^* \rangle, \quad \lambda \in \varrho(A_0).$$

Prove that if $\lambda \in \varrho(A_0)$ and $\psi^\odot \in X^\odot$, then

$$A^*\psi^\odot = \lambda\psi^\odot$$

if and only if

$$\psi^\odot = \sum_{i=1}^{N} d_i r_i^\odot(\lambda),$$

where the coefficients d_i are the components of a row vector d satisfying

$$dM(\lambda) = d.$$

4. Prove that there exists a matrix-valued function $k \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{N\times N})$ such that

$$\left\langle j^{-1}\left(\int_0^t T_0^{\odot *}(t-\tau) r_j^{\odot *}\eta(\tau)d\tau\right), r_i^*\right\rangle = \int_0^t k_{ij}(t-\tau)\eta(\tau)d\tau$$

for all $\eta \in L^1_{\text{loc}}(\mathbb{R}_+)$ and deduce that the Laplace transform $\widehat{k}$ of k is

$$\widehat{k}(\lambda) = M(\lambda), \quad \lambda \in \varrho(A_0).$$

5. During the lectures it was shown that the integrated semigroups defined on $X^{\odot *}$ by

$$\begin{aligned} W_0(t)\varphi^{\odot *} &= \int_0^t T_0^{\odot *}(s)\varphi^{\odot *} ds \\ W(t)\varphi^{\odot *} &= \int_0^t T^{\odot *}(s)\varphi^{\odot *} ds \end{aligned}$$

satisfy

$$W(t)\varphi^{\odot *} = W_0(t)\varphi^{\odot *} + \int_0^t T_0^{\odot *}(s) B W(t)\varphi^{\odot *} ds.$$

Use this to show that the solution u of

$$u(t) = T_0(t)\varphi + \int_0^t T_0^{\odot *}(t-s)(Bu(s) + f(s))ds$$

is given by

$$u(t) = T(t)\varphi + \int_0^t T^{\odot *}(t-s) f(s) ds.$$