Department of Mathematics and Statistics
Semigroups and Delay Equations
Exercises 5
21.2.2014

1. Prove that every subadditive function $p : (0, \infty) \to \mathbb{R}$ satisfies
$$\lim_{t\to\infty} \frac{p(t)}{t} = \inf_{t>0} \frac{p(t)}{t}.$$

2. Let T be a C_0-semigroup on the Banach space X. Show that for every $x \in X$ one has
$$\omega(x; T) := \inf \left\{\omega \in \mathbb{R} : \exists M \geq 1, \|T(t)x\| \leq Me^{\omega t}\ t \geq 0\right\} = \limsup_{t\to\infty} \frac{1}{t} \log\|T(t)x\|.$$

3. Let A be the infinitesimal generator of a C_0-semigroup T. Show that the spectral bound $s(A)$ is the abscissa of holomorphy of the Laplace transform
$$\lambda \mapsto \int_0^\infty e^{-\lambda t} T(t)dt$$
and that
$$\omega_1(A) := \sup \{\omega(x; T) : x \in \mathcal{D}(A)\}$$
is the abscissa of convergence.

4. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be continuos and assume that $\int_0^\infty F(t)dt < \infty$. Show that if there exist $m \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $F(t + s) \leq mF(s)$ for all $s \geq 0$ and $t \in [0, n]$, then $\lim_{s\to\infty} F(s) = 0$.

5. Let X be the Hilbert space of all square-summable sequences $x = \{x_1, x_2, \ldots\}$ with $x_n \in \mathbb{C}^n$ and let A_n be the $n \times n$-matrix
$$\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$
Define $T(t) : X \to X$ by
$$(T(t)x)_n = e^{2\pi int} e^{tA_n} x_n.$$
Show that T is a a C_0-semigroup and determine its generator. Determine $s(A), \omega_1(A)$ and $\omega_0(T)$.