Department of Mathematics and Statistics
Semigroups and Delay Equations
Exercises 4
14.2.2014

1. Let $T$ be a $C_0$-semigroup on a Banach space $X$. Show that
$$T^*(t)x^* = (\lambda I - A^*)\, T^{\odot}(t) R(\lambda, A^*)\, x^*$$
for all $x^* \in X^*$ and all $t \geq 0$.

2. Let $X$ be a Banach space and let $f : [a, b] \to X^*$ be continuous from $[a, b]$ to $X^*$ equipped with the weak*-topology. Show that if $T$ is a bounded linear operator on $X$, then
$$T^* \int_a^b f(s)ds = \int_a^b T^* f(s)ds.$$
If $S$ is a bounded linear operator on $X^*$, is it necessarily true that
$$S \int_a^b f(s)ds = \int_a^b Sf(s)ds \ ?$$

3. Let $A$ be a linear operator in the Banach space $X$. Show that
$$AR(\lambda, A)x = R(\lambda, A)Ax$$
for all $x \in \mathcal{D}(A)$ and all $\lambda \in \varrho(A)$.

4. Show that the spectrum of a bounded linear operator on a Banach space is a nonempty compact subset of the complex plane.

5. Let $T$ be a bounded linear operator from the Banach space $X$ to the Banach space $Y$. Prove that the following assertions are equivalent:

   (i) $T$ is a compact operator,

   (ii) If $U \subset X$ is the open unit ball, then $T(U)$ is totally bounded in $Y$.

   (iii) Each bounded sequence $x_n$ in $X$ has a subsequence $x_{n_i}$ such that $Tx_{n_i}$ converges in $Y$.