Department of Mathematics and Statistics
Semigroups and Delay Equations
Exercises 3
7.2.2014

1. Let A be the infinitesimal generator of a strongly continuous semigroup T with growth bound ω_0, that is, $\omega_0 := \inf\{\omega \in \mathbb{R} : \exists M \geq 1, \|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0\}$. Show that $\varrho(A^*)$ contains the right half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0\}$ and that

$$R(\lambda, A^*) = R(\lambda, A)^*$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$.

2. Let X be a Banach space and let $f : [a, b] \to X^*$ be continuous from $[a, b]$ to X^* equipped with the weak*-topology. Show that the mapping

$$x \mapsto \int_a^b \langle x, f(s)\rangle ds$$

defines a continuous linear functional on X, that is, an element of X^*. This element is denoted $\int_a^b f(s)ds$ and is called the weak* Riemann integral of f. (Hint: Banach-Steinhaus).

3. Let A be the infinitesimal generator of a strongly continuous semigroup T on the Banach space X. Prove that X is $\odot$- reflexive with respect to T if and only if $R(\lambda, A)$ is a compact operator in the weak topology on X determined by $X^\odot$, that is, the weakest topology on X making the functionals $x \mapsto \langle x, x^\odot\rangle$ contionuous for all $x^\odot \in X^\odot$.

4. Let A be the infinitesimal generator of a strongly continuous semigroup T on the Banach space X. Prove that X is $\odot$- reflexive with respect to T if and only if $X^\odot$ is $\odot$- reflexive with respect to $T^\odot$.