Department of Mathematics and Statistics
Semigroups and Delay Equations
Exercises 2
31.1.2014

1. Let A be a closed linear operator in the Banach space X and let $f : [a, b] \to X$ be continuous with $f([a, b]) \subset \mathcal{D}(A)$. Assume also that $t \mapsto Af(t)$ is continuous from $[a, b]$ to X. Show that

$$\int_a^b f(t)dt \in \mathcal{D}(A) \quad \text{and} \quad A\int_a^b f(t)dt = \int_a^b Af(t)dt.$$

2. Let X be a normed space and let Y be a Banach space. Let A_n be a sequence of bounded linear operators from X to Y such that

   (a) $\|A_n\| \leq M < \infty$ for all $n \in \mathbb{N}$,

   (b) There exists a dense subset E of X such that $A_n x$ converges to an element of X for all $x \in E$.

   Show that

   (i) The sequence $A_n x$ converges for all $x \in X$,

   (ii) $Ax := \lim_{n\to\infty} A_n x,\ x \in X$ defines a bounded linear operator A from X to Y.

3. Let A be a dissipative linear operator in X, that is,

$$\|(\lambda I - A)x\| \geq \lambda\|x\| \quad \text{for all } x \in \mathcal{D}(A),\ \lambda > 0.$$

   Show that the following conditions are equivalent

   (i) $\mathcal{R}(\lambda I - A) = X$ for all $\lambda > 0$,

   (ii) There exists a $\lambda > 0$ such that $\mathcal{R}(\lambda I - A) = X$.

   Dissipative operators satisfying the equivalent conditions (i) and (ii) are called *m-dissipative.*

4. Let $X = C[0, 1]$ and define

$$Af = -f', \quad f \in \mathcal{D}(A) := \left\{f \in C^1[0, 1] : f(0) = 0\right\}.$$

   Show that A is dissipative but not densely defined.

5. The spectrum $\sigma(A)$ of a linear operator in a Banach space X can be subdivided into three mutually disjoint sets. For every $\lambda \in \sigma(A)$ precisely one of the following conditions is satisfied.

   (i) $\lambda I - A$ is not injective. λ is then called an *eigenvalue* of A and is said to belong to the *point spectrum* $P\sigma(A)$ of A.

(ii) $\lambda I - A$ is injective, but its range is not dense in X. λ is then said to belong to the *residual spectrum* $R\sigma(A)$ of A.

(iii) $\lambda I - A$ is injective with dense range but the inverse is not bounded. λ is then said to belong to the *contiunuous spectrum* $C\sigma(A)$ of A.

Let X be one of the spaces $BUC(\mathbb{R})$, $L^p(\mathbb{R})$, $(1 \leq p < \infty)$ and define

$$Af = -f', \quad f \in \mathcal{D}(A) := \{f \in X : f \text{ is locally absolutely continuous and } f' \in X\}\,.$$

Determine the point, residual and continuous spectra of A and show that these depend on the space X. Show that A satisfies the conditions of the Hille-Yosida theorem and conclude that A generates a strongly continuous contraction semigroup on X.