Department of Mathematics and Statistics
Semigroups and Delay Equations
Exercises 1
24.1.2014

1. In the literature there occur different definitions of resolvent set and resolvent operator. During the lectures the following definition was presented:

   **Definition 1** Let $A : \mathcal{D}(A) \to X$ be a linear operator in the Banach space $X$. The *resolvent set* $\varrho(A)$ of $A$ consists of all $\lambda \in \mathbb{C}$ for which

   (i) $\lambda I - A$ is an injection,
   (ii) $\overline{\mathcal{R}(\lambda I - A)} = X$,
   (iii) $(\lambda I - A)^{-1} : \mathcal{R}(\lambda I - A) \to X$ is a bounded linear operator.

   For all $\lambda \in \varrho(A)$ the *resolvent operator* $R(\lambda, A)$ of $A$ is the unique extension of $(\lambda I - A)^{-1}$ to a bounded linear operator on $X$.

   Here is another popular definition of resolvent:

   **Definition 2** Let $A : \mathcal{D}(A) \to X$ be a linear operator in the Banach space $X$. The *resolvent set* $\varrho(A)$ of $A$ consists of all $\lambda \in \mathbb{C}$ for which

   (i) $\lambda I - A$ is a bijection $\mathcal{D}(A) \to X$,
   (ii) $(\lambda I - A)^{-1}$ is a bounded linear operator on $X$.

   For all $\lambda \in \varrho(A)$ the *resolvent operator* of $A$ is $R(\lambda, A) = (\lambda I - A)^{-1}$.

   Prove that

   (a) If $A$ is a closed operator, then the two definitions are equivalent but that in general they are not.
   (b) If $\varrho(A) \neq \emptyset$ according to Definition 2, then $A$ is closed. The converse is false: A closed operator may have empty resolvent set.

2. Let $r > 0$ and let $X = C_0[-r, 0)$ be the Banach space of all continuous functions on $[-r, 0]$ that vanish at 0. Define for all $f \in X$

   $$(T(t)f)(\theta) = \begin{cases} f(\theta + t) & \text{if} \quad 0 \leq t \leq -\theta, \\ 0 & \text{if} \quad -\theta < t, \end{cases} \qquad \theta \in [-r, 0].$$

   Show that $T$ is a $C_0$-semigroup on $X$, determine its infinitesimal generator $A$ and find $\varrho(A)$ and $R(\lambda, A)$. Do the same in the case $X = L^p[-r, 0]$ $(1 \leq p < \infty)$.

3. Let $X$ be either $L^p(\mathbb{R})$ $(1 \leq p < \infty)$ or $BUC(\mathbb{R})$ and define for all $f \in X$

   $$(T(t)f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-y^2/2t} f(x - y) dy \quad y \in \mathbb{R}, \quad t > 0$$

   and put $T(0) = I$.

   Prove that $T$ is a $C_0$-semigroup on $X$, determine its infinitesimal generator $A$ and find $\varrho(A)$ and $R(\lambda, A)$.

4. Let $A$ be the infinitesimal generator of a $C_0$-semigroup $T$ satisfying $\|T(t)\| \leq M$ for $t \geq 0$. Show that if $x \in \mathcal{D}(A^2)$, then

$$\|Ax\|^2 \leq 4M^2 \|A^2 x\| \|x\|.$$

5. Apply Exercise 4 to the semigroup of left translations on $BUC(\mathbb{R})$ to prove *Landau's inequality*

$$\left(\sup_{x\in\mathbb{R}} |f'(x)|\right)^2 \leq 4 \sup_{x\in\mathbb{R}} |f''(x)| \ \sup_{x\in\mathbb{R}} |f(x)|.$$