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Perturbation Theory for Dual Semigroups

III. Nonlinear Lipschitz Continuous Perturbations in the Sun-Reflexive Case

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We consider nonlinear Lipschitz perturbations of the infinitesimal generator of a linear C_0-semigroup on a non-reflexive Banach space. It is allowed that the perturbation maps the space into a bigger space which arises in a natural way when considering dual semigroups. Using a generalized variation - of - constants formula we show that the perturbed operator generates a strongly continuous nonlinear semigroup. We study regularity properties of this semigroup and prove the principle of linearized stability.

Keywords & Phrases: strongly continuous semigroup, dual semigroup, weakly * continuous semigroup, Favard class, weak * Riemann integral, variation - of - constants formula, inhomogeneous initial value problem, nonlinear Lipschitz continuous perturbation, semilinear equation, principle of linearized stability, Volterra integral equation.
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1. Introduction

The aim of this paper is to demonstrate how perturbation theory for dual semigroups can be used to write certain nonlinear evolution equations as abstract *semilinear* equations. Becoming familiar with our duality framework (the game of suns and stars) requires, as always, a certain investment of time and energy of the reader. The refund includes, among other things, 1) the possibility to use standard techniques to prove local stability and bifurcation results, and 2) the unification of theories for various equations, like delay equations [13, 10] and age dependent population equations [19, 2].

In this paper we concentrate on the principle of linearized stability while postponing the derivation of results concerning invariant manifolds to future papers. As a prelude to semilinear problems we present in Section 2 some results about inhomogeneous linear equations which are, in our opinion, interesting by themselves. We hope to return to this topic too at some later time.

In this paper we try to convey the flavour of our recent work which was largely motivated by physiologically structured population models [17]. It should be noted that Desch, Schappacher and co-workers [7, 9] as well as Greiner [11, 12] and Kellermann [16] developed alternative approaches for the study of this kind of problems.

Consider a semigroup $T_0(t)$ of bounded linear operators acting on some Banach space X. Assume that $T_0(t)$ is strongly continuous, i.e. the orbit $t \mapsto T_0(t)x$ is continuous for each initial value $x \in X$. If X is non-reflexive, the dual semigroup $T_0^*(t)$ need not have the same property. All one can say in general is that the orbits are continuous in the weak * topology. At the level of the generators this phenomenon is reflected in the fact that A_0^*, the adjoint of the infinitesimal generator A_0 of $T_0(t)$, need not have dense domain and that A_0^* is the weak * generator of $T_0^*(t)$ (see Butzer & Berens [1]).

The basic idea now is to take advantage of this phenomenon and to construct a second dual space for X based on the behaviour of the semigroup for $t \downarrow 0$ (or, equivalently, the behaviour of the resolvent of the generator for $\lambda \to \infty$).

To make this more precise we introduce

$$X^{\odot} = \{x^* \in X^* : \lim_{t \downarrow 0} \|T_0^*(t)x^* - x^*\| = 0\}.$$

The space $X^{\odot}$ is the maximal invariant subspace of X^* on which $T_0^*(t)$ is strongly continuous. Moreover, it is known that $X^{\odot} = \overline{\mathfrak{D}(A_0^*)}$ (see [1, 14]). The restriction $T_0^{\odot}(t)$ of $T_0^*(t)$ to $X^{\odot}$ is strongly continuous and is generated by $A_0^{\odot}$, the part of A_0^* in $X^{\odot}$ (that is, $\mathfrak{D}(A_0^{\odot}) = \{x^{\odot} \in X^{\odot} : x^{\odot} \in \mathfrak{D}(A_0^*)$ and $A_0^* x^{\odot} \in X^{\odot}\}$). So on $X^{\odot}$ we now have the same situation as we had on X at the start. Consequently we can play the same game once more. In self-explaining notation we obtain $X^{\odot *}, T_0^{\odot *}(t), A_0^{\odot *}$ and $X^{\odot\odot}, T_0^{\odot\odot}(t)$ and $A_0^{\odot\odot}$. The pairing between elements of X and $X^{\odot}$ defines an embedding j of X into $X^{\odot *}$. Note that $j(X) \subset X^{\odot\odot}$.

Definition. X is $\odot$-reflexive (pronounce: sun-reflexive) with respect to $T_0(t)$ if and only if $j(X) = X^{\odot\odot}$.

In this paper we shall restrict our attention to the $\odot$-reflexive case. Moreover, we shall in the following identify X with its embedding $j(X)$ and omit the symbol j.

Integrals of the form $\int_0^t T_0^{\odot *}(t-\tau)h(\tau)d\tau$, where h is a given weakly * continuous $X^{\odot *}$-valued function, are now defined as weak * Riemann integrals. This means that the integral is by definition the continuous linear functional on $X^{\odot}$ which assigns to an arbitrary $x^{\odot} \in X^{\odot}$ the value

$$\int_0^t \langle h(\tau), T_0^{\odot}(t-\tau)x^{\odot} \rangle d\tau.$$

Section 2 provides further information on these integrals.

In our previous paper [2] (also see [4]) we showed that any bounded linear operator

$$B : X \to X^{\odot *}$$

defines via solutions of the variation - of - constants equation

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(t-\tau)BT(\tau)x d\tau$$

a "perturbed" strongly continuous semigroup $T(t)$ on X such that
(i) the spaces $X^{\odot}$ and $X^{\odot\odot}$ do not change, so in particular they are independent of B,
(ii) on the "big" spaces X^* and $X^{\odot *}$ the domains of the weak * generators remain unchanged, i.e. $\mathfrak{D}(A^*)=\mathfrak{D}(A_0^*)$, and $\mathfrak{D}(A^{\odot *})=\mathfrak{D}(A_0^{\odot *})$,
(iii) $A^{\odot *}=A_0^{\odot *}+B$ and $A^*=A_0^*+B^*$.
Since A is the part of $A^{\odot *}$ in X the domain $\mathfrak{D}(A)$ in the "small" space X may depend on B, even to the extent that all information about B is contained in $\mathfrak{D}(A)$, the action of A_0 and A being the same (this actually happens for functional differential equations and age-dependent population equations). Thus we see quite clearly how the duality frame-work is exploited to handle the equation $dx/dt=Ax$ by perturbation methods. In the following we shall consider nonlinear perturbations

$$F : X \rightarrow X^{\odot *}$$

and use basically the same approach (see [3] for linear time-dependent perturbations $B(t)$).
In conclusion of this introduction we mention a useful fact which follows from dual semigroup theory. The so-called Favard class of $T_0(t)$ is defined by

$$\mathrm{Fav}(T_0) = \{x \in X : \limsup_{t \downarrow 0} \frac{1}{t} \|T(t)x - x\| < \infty\}.$$

It is known ([1, 4]) that Fav $(T_0)=\mathfrak{D}(A_0^{\odot *})$. Moreover, Fav (T_0) is precisely the set of initial data which yield Lipschitz continuous orbits. It is this latter non-local characterization which immediately guarantees the invariance and which, therefore, is very useful in nonlinear situations.
Section 2 deals with linear inhomogeneous equations. In Section 3 we derive results about the existence, uniqueness and differentiability of solutions of semilinear equations and Section 4 is devoted to the principle of linearized stability. In Section 5, finally, we show how to obtain solutions corresponding to initial data in the 'big' space $X^{\odot *}$ if the nonlinear perturbation is somewhat special (in particular it must haven finite dimensional range). This is achieved by solving a nonlinear Volterra convolution equation in L_∞.

2. The linear inhomogeneous initial value problem

Let $T_0(t)$ be a strongly continuous semigroup on the (non-reflexive) Banach space X, and assume that X is sun-reflexive with respect to A_0, the generator of $T_0(t)$. There exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that the following estimate holds:

$$\|T_0(t)\| \leq M e^{\omega t}, \quad t \geq 0.$$

In this section we consider the inhomogeneous initial value problem

$$\begin{cases} \frac{du}{dt}(t) = A_0^{\odot *}u(t) + f(t) \ , \ 0<t<T, \\ u(0) = x \in X, \end{cases} \tag{2.1}$$

where $f:[0,T] \rightarrow X^{\odot *}$ is a given weakly * continuous function. We define the so-called mild solution to (2.1) by

$$u(t) = T_0(t)x + \int_0^t T_0^{\odot *}(t-s)f(s)ds, \quad 0 \leq t \leq T, \tag{2.2}$$

where the integral is a weak * Riemann integral. We also define

$$v(t) = \int_0^t T_0^{\odot *}(t-s)f(s)ds, \quad 0 \leqslant t \leqslant T. \tag{2.3}$$

We will successively consider what can be said about v if f is weakly * continuous, norm continuous, Lipschitz continuous, and continuously differentiable.

PROPOSITION 2.1. *If f is weakly * continuous then v is weakly * continuous with values in $X^{\odot *}$. If f is norm continuous then v is norm continuous as well and takes values in X. In both cases the following estimate holds:*

$$\|v(t)\| \leqslant \frac{M}{\omega}(e^{\omega t}-1) \cdot \sup_{0 \leqslant s \leqslant t} \|f(s)\| .$$

A proof of this result can be found in [2].

PROPOSITION 2.2. *Let f be (locally) Lipschitz continuous. Then v is weakly * continuously differentiable (in particular, v is Lipschitz continuous), v takes values in $\mathfrak{D}(A_0^{\odot *})$, and $w^* - \frac{dv}{dt}(t) = A_0^{\odot *}v(t) + f(t)$. Here $w^* - \frac{d}{dt}$ denotes the weak * derivative.*

PROOF. To show that v takes values in $\mathfrak{D}(A_0^{\odot *})$ we use that $\mathfrak{D}(A_0^{\odot *}) = \mathrm{Fav}(T_0)$. Let $t \geqslant 0$ and $h > 0$. Then

$$\frac{1}{h}\|T_0(h)v(t) - v(t)\| \leqslant \frac{1}{h}\| \int_t^{t+h} T_0^{\odot *}(s)f(t+h-s)ds \|$$

$$+\frac{1}{h}\| \int_h^t T_0^{\odot *}(s)\{f(t+h-s) - f(t-s)\}ds \| + \frac{1}{h}\| \int_0^h T_0^{\odot *}(s)f(t-s)ds \| ,$$

which stays bounded as $h \downarrow 0$. It also follows from this expression that $\|A_0^{\odot *}v(t)\|$ is uniformly bounded on compact t-intervals. Now we show that $t \to A_0^{\odot *}v(t)$ is weakly * continuous. First take $x^{\odot} \in \mathfrak{D}(A_0^{\odot})$. Then

$$<A_0^{\odot *}v(t), x^{\odot}> = <v(t), A_0^{\odot}x^{\odot}>$$

is continuous. The continuity of $<A_0^{\odot *}v(\cdot), x^{\odot}>$ for arbitrary $x^{\odot} \in X^{\odot}$ now easily follows from the observation that $\mathfrak{D}(A_0^{\odot})$ lies dense in $X^{\odot}$ and that $\|A_0^{\odot *}v(t)\|$ is bounded on compact t-intervals. Finally we show that $v(t) = \int_0^t \{A_0^{\odot *}v(s) + f(s)\}ds$, which completes the proof. Let $x^{\odot} \in \mathfrak{D}(A_0^{\odot})$.

$$<v(t), x^{\odot}> - \int_0^t <f(s), x^{\odot}>ds = \int_0^t <f(s), T^{\odot}(t-s)x^{\odot} - x^{\odot}>ds =$$

$$= \int_0^t <f(s), \int_0^{t-s} T_0^{\odot}(\sigma)A_0^{\odot}x^{\odot}d\sigma>ds = \int_0^t <f(s), \int_s^t T_0^{\odot}(\sigma - s)A_0^{\odot}x d\sigma>ds =$$

$$= \int_0^t \int_s^t <T_0^{\odot *}(\sigma - s)f(s), A_0^{\odot}x^{\odot}>d\sigma ds = \int_0^t \int_0^{\sigma} <T_0^{\odot *}(\sigma - s)f(s), A_0^{\odot}x^{\odot}>ds d\sigma =$$

$$= \int_0^t <v(\sigma), A_0^{\odot}x^{\odot}>d\sigma = \int_0^t <A_0^{\odot *}v(\sigma), x^{\odot}>d\sigma .$$

Since $\mathfrak{D}(A_0^{\odot})$ is dense in $X^{\odot}$ the same identity holds for arbitrary $x^{\odot} \in X^{\odot}$ □

PROPOSITION 2.3. *Assume that f is continuously differentiable on $[0,T)$ and that $f(0) \in X$. Then v is* *continuously differentiable on* $[0,T)$ *and takes values in* $\mathfrak{D}(A_0^{\odot *})$. *Furthermore*

$$v'(t)=A_0^{\odot *}v(t)+f(t)=T_0(t)f(0)+\int_0^t T_0^{\odot *}(t-s)f'(s)ds\in X\,.$$

PROOF. From Proposition 2.2 we know that $v(t)\in\mathfrak{D}(A_0^{\odot *})$, $0\leq t\leq T$. We compute $A_0^{\odot *}v(t)$ explicitly.

$$\frac{1}{h}(T_0(h)-I)v(t)=\frac{1}{h}\int_{-h}^{0} T_0^{\odot *}(t-s)f(s+h)ds$$

$$-\frac{1}{h}\int_{t-h}^{t} T_0^{\odot *}(t-s)f(s+h)ds+\frac{1}{h}\int_0^t T_0^{\odot *}(t-s)\{f(s+h)-f(s)\}ds$$

$$\to T_0^{\odot *}(t)f(0)-f(t)+\int_0^t T_0^{\odot *}(t-s)f'(s)ds\,,$$

in the weak * topology, as $h\downarrow 0$, and by definition this limit equals $A_0^{\odot *}v(t)$. Next we show that v is continuously differentiable.

$$\frac{1}{h}(v(t+h)-v(t))=\frac{1}{h}\int_{-h}^{0} T_0^{\odot *}(t-s)f(s+h)ds+\frac{1}{h}\int_0^t T_0^{\odot *}(t-s)\{f(s+h)-f(s)\}ds=$$

$$T_0(t)\frac{1}{h}\int_0^h T_0(s)f(0)ds+T_0(t)\frac{1}{h}\int_0^h T_0^{\odot *}(s)\{f(h-s)-f(0)\}ds+$$

$$\int_0^t T_0^{\odot *}(t-s)f'(s)ds+\int_0^t T_0^{\odot *}(t-s)\{\frac{f(s+h)-f(s)}{h}-f'(s)\}ds$$

$$\to T_0(t)f(0)+\int_0^t T_0^{\odot *}(t-s)f'(s)ds$$

in norm as $h\downarrow 0$. So v is right-differentiable and the right-derivative is continuous. Therefore $v(t)$ is continuously differentiable and

$$v'(t)=T_0(t)f(0)+\int_0^t T_0^{\odot *}(t-s)f'(s)ds=A_0^{\odot *}v(t)+f(t)\,.$$

This proves the proposition. □

COROLLARY 2.4. *Assume that* f *is continuously differentiable. Let* $x\in\mathfrak{D}(A_0^{\odot *})$ *and* $A_0^{\odot *}x+f(0)\in X$. *Then* $u(t)$ *given by (2.2) is continuously differentiable and takes values in* $\mathfrak{D}(A_0^{\odot *})$. *Furthermore* $A_0^{\odot *}u(t)+f(t)\in X$, $0\leq t\leq T$, *and* u *satisfies (2.1). In other words, (2.1) admits a classical solution.*

PROOF.
Define $\tilde{f}(t)=A_0^{\odot *}x+f(t)$, and let

$$\tilde{v}(t)=\int_0^t T_0^{\odot *}(t-s)\tilde{f}(t)dt=\int_0^t T_0^{\odot *}(t-s)A_0^{\odot *}x+v(t)$$

$$=T_0(t)x-x+v(t)=u(t)-x\,,$$

where v, u are given by (2.3) and (2.2) respectively. Now $\tilde{f}$ satisfies the assumption of Proposition 2.3, hence $\tilde{v}$ is C^1 and $\tilde{v}'(t)=A_0^{\odot *}\tilde{v}(t)+\tilde{f}(t)$, or equivalently, u is C^1 and $u'(t)=A_0^{\odot *}u(t)+f(t)$. □

We refer to DA PRATO and SINESTRARI [6] for a more general result. Also see KELLERMANN [15].

When studying the equation $du/dt = A_0^{\odot *}u + Bu + f$ with B a given bounded linear operator from X into $X^{\odot *}$, we may either consider $Bu+f$ as a perturbation and solve a variation-of-constants integral equation involving $T_0^{\odot *}(t)$ or first define the semigroup $T(t)$ on X generated by the part of $A_0^{\odot *}+B$ in X and write down the solution explicitly as a variation-of-constants formula involving $T(t)$. Of course, both procedures should define the same solution. That they actually do so is the content of the following proposition.

PROPOSITION 2.5. *Let $B: X \to X^{\odot *}$ be a bounded linear operator and let $T(t)$ be the C_0-semigroup on X generated by A, the part of $A_0^{\odot *}+B$ in X. Let $x \in X$ and let $f:[0,T] \to X^{\odot *}$ be an arbitrary continuous function. Let $u(t)$ be the norm-continuous solution of the integral equation*

$$u(t) = T_0(t)x + \int_0^t T_0^{\odot *}(t-s)\{Bu(s)+f(s)\}ds, \quad 0 \le t \le T, \tag{2.4}$$

then

$$u(t) = T(t)x + \int_0^t T^{\odot *}(t-s)f(s)ds.$$

To prove this proposition we need a lemma which is of interest by itself. Let A,B be as in Proposition 2.5 and define the integrated semigroups (see [15]) $W_0(t), W(t): X^{\odot *} \to X$ by

$$W_0(t)x^{\odot *} = \int_0^t T_0^{\odot *}(s)x^{\odot *}ds$$

$$W(t)x^{\odot *} = \int_0^t T^{\odot *}(s)x^{\odot *}ds.$$

LEMMA 2.6. *For every $x^{\odot *} \in X^{\odot *}$ and $t \ge 0$,*

$$W(t)x^{\odot *} = W_0(t)x^{\odot *} + \int_0^t T_0^{\odot *}(t-s)BW(s)x^{\odot *}ds.$$

PROOF. From the variation-of-constants formula

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(s)BT(t-s)xds$$

it follows that for every $x \in X$, $x^{\odot} \in X^{\odot}$,

$$<\int_0^t T(\tau)xd\tau, x^{\odot}> = <\int_0^t T_0(\tau)xd\tau, x^{\odot}> + \int_0^t \int_0^\tau <T(\tau-s)x, B^* T_0^{\odot}(s)x^{\odot}>dsd\tau$$

where $B^*: X^{\odot} \to X^*$ is the restriction of the dual operator of B to $X^{\odot}$. By changing the order of integration in the last integral we find that

$$<W(t)x, x^{\odot}> = <W_0(t)x, x^{\odot}> + \int_0^t <W(t-s)x, B^* T_0^{\odot}(s)x^{\odot}>ds.$$

Let $x^{\odot *} \in X^{\odot *}$ and let $\{x_n\}$ be a sequence in X converging to $x^{\odot *}$ in the weak * sense. It is easily seen that $W_0(t)^* = \int_0^t T_0^*(\tau)d\tau$ and $W(t)^* = \int_0^t T^*(\tau)d\tau$ and that they map X^* into $X^{\odot}$ and consequently that $W_0(t)x_n \to W_0(t)x^{\odot *}$ and $W(t)x_n \to W(t)x^{\odot *}$ with respect to the weak topology. By the dominated convergence theorem it follows that

$$\int_0^t <W(t-s)x_n, B^* T_0^{\odot}(s)x^{\odot}>ds \to \int_0^t <W(t-s)x^{\odot *}, B^* T_0^{\odot}(s)x^{\odot}>ds.$$

From this, the result follows immediately. □

Proof of Proposition 2.5. It is easy to show that (2.4) has a unique continuous solution. Therefore we have to check that substitution of $u(t)=T(t)x+\int_0^t T^{\odot *}(t-s)f(s)ds$ into equation (2.4) gives an identity. This amounts to verifying that

$$\int_0^t T^{\odot *}(t-s)f(s)ds = \int_0^t T_0^{\odot *}(t-s)f(s)ds + \int_0^t T_0^{\odot *}(s)B\{\int_0^{t-s} T^{\odot *}(t-s-\sigma)f(\sigma)d\sigma\}ds. \qquad (*)$$

In order to achieve this we integrate both sides of the equation from 0 to t. Clearly

$$\int_0^t\{\int_0^\tau T^{\odot *}(\tau-s)f(s)ds\}d\tau = \int_0^t W(t-s)f(s)ds,$$

and we find a similar identity for the integral of the first term at the right-hand-side. We evaluate the integral of the last term by pairing it with an element $x^{\odot}\in X^{\odot}$.

$$<\int_0^t\{\int_0^\tau T_0^{\odot *}(s)B\{\int_0^{\tau-s} T^{\odot *}(\tau-s-\sigma)f(\sigma)d\sigma\}ds\}d\tau, x^{\odot}> =$$

$$\int_0^t\int_0^\tau <\int_0^{\tau-s} T^{\odot *}(\tau-s-\sigma)f(\sigma)d\sigma, B^* T_0^{\odot}(s)x^{\odot}>ds d\tau =$$

$$\int_0^t\int_s^t <\int_0^{\tau-s} T^{\odot *}(\tau-s-\sigma)f(\sigma)d\sigma, B^* T_0^{\odot}(s)x^{\odot}>d\tau ds =$$

$$\int_0^t <\int_s^t \int_0^{\tau-s} T^{\odot *}(\tau-s-\sigma)f(\sigma)d\sigma d\tau, B^* T_0^{\odot}(s)x^{\odot}>ds =$$

$$\int_0^t <\int_0^{t-s}\int_0^{t-s-\sigma} T^{\odot *}(\tau)f(\sigma)d\tau d\sigma, B^* T_0^{\odot}(s)x^{\odot}>ds =$$

$$\int_0^t <\int_0^{t-s} W(t-s-\sigma)f(\sigma)d\sigma, B^* T_0^{\odot}(s)x^{\odot}>ds =$$

$$<\int_0^t T_0^{\odot *}(s)B\{\int_0^{t-s} W(t-s-\sigma)f(\sigma)d\sigma\}ds, x^{\odot}> =$$

$$<\int_0^t\{\int_0^{t-\sigma} T_0^{\odot *}(t-\sigma-\tau)BW(\tau)\}d\tau f(\sigma)d\sigma, x^{\odot}> =$$

$$<\int_0^t W(t-\sigma)f(\sigma)d\sigma - \int_0^t W_0(t-\sigma)f(\sigma)d\sigma, x^{\odot}>$$

because of Lemma 2.6. Hence, the integral from 0 to t of (*) yields an identity. Moreover (*) becomes an identity for $t=0$, and from this the result follows. □

3. The semilinear equation

Let $F:X\rightarrow X^{\odot *}$ be a (nonlinear) function which is globally Lipschitz continuous with Lipschitz constant L, i.e.

$$\|F(x) - F(y)\| \leqslant L\|x-y\|, \quad x,y\in X.$$

We assume *global* Lipschitz continuity for ease of formulation. Appropriate variants of our results can be formulated when F is only locally Lipschitz continuous but our aim here is to explain the essence of our approach in as simple a setting as possible.

In this section we examine the initial value problem

$$\frac{du}{dt}(t) = A_0^{\odot *}u(t) + F(u(t)), \quad t>0$$
$$u(0) = x. \tag{3.1}$$

Note that even when orbits $t\rightarrow u(t;x)$ in X are considered, the differential equation should be interpreted as an identity of $X^{\odot *}$-valued functions. We can reformulate (3.1) as an integral equation:

$$u(t) = T_0(t)x + \int_0^t T_0^{\odot *}(t-s)F(u(s))ds. \tag{3.2}$$

We shall first study this integral equation and subsequently discuss the relation with the differential equation (3.1).

If $u(\cdot)$ is a norm continuous X-valued function then $F(u(\cdot))$ is a norm continuous $X^{\odot *}$-valued function, and from Proposition 2.1 we conclude that the integral at the right-hand-side of (3.2) defines a norm continuous X-valued function. Moreover, if $u_1(\cdot)$, $u_2(\cdot)$ are two norm continuous functions, then

$$\|\int_0^t T_0^{\odot *}(t-s)F(u_1(s))ds - \int_0^t T_0^{\odot *}(t-s)F(u_2(s))ds\| \leqslant$$
$$\int_0^t Me^{\omega(t-s)}L\|u_1(s)-u_2(s)\|ds \leqslant ML\frac{e^{\omega t}-1}{\omega}\sup_{0\leqslant s\leqslant t}\|u_1(s)-u_2(s)\|.$$

Therefore, we can invoke standard contraction and continuation arguments to prove existence and uniqueness of solutions to (3.2).

Theorem 3.1. *For every $x\in X$ there exists a unique continuous solution $u(\cdot;x)$ to the integral equation (3.2). This solution has the semigroup property*

$$u(t+s;x) = u(t;u(s;x)), \quad s,t\geqslant 0.$$

Furthermore, the following estimates hold:

$$\|u(t;x)\| \leqslant (M\|x\| + \frac{M}{\omega}\|F(0)\|)e^{(\omega+ML)t}, \quad t\geqslant 0, \tag{3.3}$$

$$\|u(t;x)-u(t;y)\| \leqslant M\|x-y\|e^{(\omega+ML)t}, \quad t\geqslant 0, \tag{3.4}$$

for every $x,y\in X$.

The estimates (3.3) and (3.4) follow easily from the Gronwall lemma. We define the non-linear strongly continuous semigroup $S(t):X\rightarrow X$ by

$$S(t)x := u(t;x), \quad t\geqslant 0, \quad x\in X.$$

The remainder of this section is devoted to the question in which sense $S(t)x$ satisfies the differential equation (3.1). To this end we need some definitions. The weak * generator $A_S^{\odot *}$ (note the abuse of notation) is defined as follows: $x \in \mathfrak{D}(A_S^{\odot *})$ if $h^{-1}(S(h)x - x)$ converges with respect to the weak * topology of $X^{\odot *}$ to some element $y^{\odot *} \in X^{\odot *}$ as $h \downarrow 0$ and in this case $A_S^{\odot *} x = y^{\odot *}$. The strong generator is defined as usual. Finally, we define the Favard class Fav (S) as in the linear case:

$$\mathrm{Fav}(S) = \{x \in X : \limsup_{h \downarrow 0} \frac{1}{h} \|S(h)x - x\| < \infty\}.$$

We can now prove the following important result.

THEOREM 3.2. *$\mathfrak{D}(A_S^{\odot *}) = \mathrm{Fav}(S) = \mathrm{Fav}(T_0) = \mathfrak{D}(A_0^{\odot *})$, and $A_S^{\odot *} x = A_0^{\odot *} x + F(x)$ for $x \in \mathfrak{D}(A_0^{\odot *})$.*

PROOF. For any $x \in X$, the expression $h^{-1} \int_0^h T_0^{\odot *}(h-s) F(S(s)x) ds$ converges towards $F(x)$ with respect to the weak * topology as $h \downarrow 0$. This implies that $\mathfrak{D}(A_S^{\odot *}) = \mathfrak{D}(A_0^{\odot *})$ and that $A_S^{\odot *} x = A_0^{\odot *} x + F(x)$ for $x \in \mathfrak{D}(A_0^{\odot *})$. But we may also conclude that $\|h^{-1} \int_0^h T_0^{\odot *}(h-s) F(S(s)x) ds\|$ remains bounded as $h \downarrow 0$, which yields that Fav (S) = Fav (T_0). From the linear theory we know that Fav $(T_0) = \mathfrak{D}(A_0^{\odot *})$ and the result is proved. □

PROPOSITION 3.3. *$x \in$ Fav (S) if and only if the orbit $t \to S(t)x$ is locally Lipschitz continuous.*

The proof of this result, which is relatively easy, can be found in a paper by CRANDALL [5], who calls Fav (S) the *generalized domain* of S. This result implies in particular that $\mathfrak{D}(A_S^{\odot *})$ is invariant under $S(t)$: namely Fav (S) is invariant under $S(t)$. Proposition 2.2 implies that $S(\cdot)x$ is the solution to the differential equation (3.1) interpreted in the weak * sense if $x \in \mathfrak{D}(A_S^{\odot *})$:

THEOREM 3.4. *If $x \in \mathfrak{D}(A_S^{\odot *})$, then $S(\cdot)x$ is weakly * continuously differentiable and*

$$w^* - \frac{du}{dt}(t) = A_0^{\odot *} u(t) + F(u(t)).$$

THEOREM 3.5. *A_S is the part of $A_S^{\odot *}$ in X, i.e. $\mathfrak{D}(A_S) = \{x \in \mathfrak{D}(A_S^{\odot *}) : A_S^{\odot *} x \in X\}$ and $A_S x = A_S^{\odot *} x$.*

PROOF. It is clear that $\mathfrak{D}(A_S) \subseteq \{x \in \mathfrak{D}(A_S^{\odot *}) : A_S^{\odot *} x \in X\}$. Conversely, assume that $x \in \mathfrak{D}(A_S^{\odot *})$ and $A_S^{\odot *} x \in X$. Then

$$\frac{1}{h}(S(h)x - x) = \frac{1}{h}(T_0(h)x - x) + \frac{1}{h}\int_0^h T_0^{\odot *}(h-s) F(S(s)x) ds =$$

$$\frac{1}{h}\int_0^h T_0^{\odot *}(h-s)\{A_0^{\odot *} x + F(x)\} ds + \frac{1}{h}\int_0^h T_0^{\odot *}(h-s)\{F(S(s)x) - F(x)\} ds.$$

The first integral can be replaced by $h^{-1} \int_0^h T_0(h-s)\{A_0^{\odot *} x + F(x)\} ds$ and converges in norm to $A_0^{\odot *} x + F(x)$, whereas the second integral converges in norm to 0. This concludes the proof. □

To conclude this section we state a result which gives sufficient conditions in order that $t \to S(t)x$ is a *classical solution* (see [18]) of the differential equation (3.1).

THEOREM 3.6. *Assume that f is continuously Fréchet differentiable. If $x \in \mathfrak{D}(A_S)$ then $u(t) := S(t)x$ is continuously differentiable and $u'(t) = A_0^{\odot *} u(t) + F(u(t))$.*

PROOF. Let $x \in \mathfrak{D}(A_S)$. Below we shall prove that $u(t) = S(t)x$ is continuously differentiable on $\mathbb{R}_+$. Then f given by $f(t) := F(u(t))$ is continuously differentiable as well, and, moreover, $A_0^{\odot *} x + f(0) = A_0^{\odot *} x + F(x) \in X$. From Corollary 2.4 we may conclude that $\tilde{u}(t)$ given by $\tilde{u}(t) = T_0(t)x + \int_0^t T_0^{\odot *}(t-s) f(s) ds = T_0(t)x + \int_0^t T_0^{\odot *}(t-s) F(u(s)) ds$ is C^1, takes values in $\mathfrak{D}(A_0^{\odot *})$

and $\tilde{u}'(t)=A_0^{\odot*}\tilde{u}(t)+f(t)$. But $\tilde{u}(t)=u(t)$ and the result is proved.

So it remains to show that $u(\cdot)\in C^1$. Define $B(t):=F'(u(t))$, $t\geq 0$. Then $B(t):X\to X^{\odot*}$ is a bounded linear operator for every $t\geq 0$ and $t\to B(t)$ is norm continuous. We also define

$$\eta(h,s) = F(u(s+h)) - F(u(s)) - B(s)(u(s+h)-u(s)).$$

Then $h^{-1}\|\eta(h,s)\|\to 0$, $h\downarrow 0$, uniformly on bounded s-intervals. In [3] we have proved that the family $A_0^{\odot*}+B(t)$ "generates" a strongly continuous evolutionary system $U(t,s)$ on X. The function $w(t):=U(t,0)A_S x$ is a solution of the integral equation

$$w(t) = T_0(t)A_S x + \int_0^t T_0^{\odot*}(t-s)B(s)w(s)ds.$$

We also define $w_h(t):=h^{-1}(u(t+h)-u(t))-w(t)$. Then

$$\begin{aligned} w_h(t) &= \frac{1}{h}(T_0(t+h)x-T_0(t)x) + \frac{1}{h}T_0(t)\int_0^h T_0^{\odot*}(h-s)F(u(s))ds \\ &+ \frac{1}{h}\int_0^t T_0^{\odot*}(t-s)\{F(u(s+h))-F(u(s))\}ds - T_0(t)A_S x \\ &- \int_0^t T_0^{\odot*}(t-s)B(s)w(s)ds \\ &= T_0(t)\{\frac{1}{h}(T_0(h)x-x) + \frac{1}{h}\int_0^h T_0^{\odot*}(h-s)F(u(s))ds - A_S x\} \\ &+ \int_0^t T_0^{\odot*}(t-s)\{B(s)(\frac{u(s+h)-u(s)}{h}) + \frac{1}{h}\eta(h,s)\}ds \\ &- \int_0^t T_0^{\odot*}(t-s)B(s)w(s)ds \\ &= T_0(t)\{\frac{1}{h}(S(h)x-x) - A_S x\} + \frac{1}{h}\int_0^t T_0^{\odot*}(t-s)\eta(h,s)ds \\ &+ \int_0^t T_0^{\odot*}(t-s)B(s)w_h(s)ds. \end{aligned}$$

Fix $t>0$. It is easy to check that

$$\|T_0(t')\{\frac{1}{h}(S(h)x-x) - A_S x\}\| + \frac{1}{h}\|\int_0^{t'} T_0^{\odot*}(t'-s)\eta(h,s)ds\| \leq \epsilon_t(h)$$

for $t'\leq t$. Here $\epsilon_t(h)$ is a function of h (for fixed t) which goes to 0 as $h\downarrow 0$. Thus we get

$$\|w_h(t')\| \leq \epsilon_t(h) + C_t\int_0^{t'}\|w_h(s)\|ds,$$

for some constant $C_t>0$. Application of Gronwall's lemma yields that

$$\|w_h(t')\| \leq \epsilon_t(h)e^{t'C_t}, \quad t'\leq t,$$

from which we deduce that $w_h(t')\to 0$, $h\downarrow 0$, for every $t'\geq 0$. This implies that $u(t)$ is right-differentiable and it's right derivative is $w(t)=U(t,0)A_S x$ which is a continuous function. Thus we have proved that $u\in C^1$. □

4. LINEARIZED STABILITY

Let $\bar{x}\in X$ be an equilibrium of the nonlinear semigroup $S(t)$, i.e.

$$S(t)\bar{x} = \bar{x}, \quad t\geqslant 0,$$

or equivalently,

$$\bar{x}\in\mathfrak{D}(A_s) \text{ and } A_S\bar{x} = 0.$$

In this section we prove the *principle of linearized stability* which says that local (in-)stability of the equilibrium $\bar{x}$ is completely determined by the spectral properties of the linearization of the nonlinear operator "$A_0^{\odot *}+F(\cdot)$" around $u=\bar{x}$.

In what follows we assume that F is continuously Fréchet differentiable at $\bar{x}$, and we denote the Fréchet derivative $DF(\bar{x})$ by B. Thus B is a bounded linear operator from X into $X^{\odot *}$. In [2] we showed that the part of $A_0^{\odot *}+B$ in X, which we denote by A, generates a linear C_0-semigroup $T(t), t\geqslant 0$, on X. It satisfies the variation-of-constants formula

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(t-s)BT(s)x ds, \quad t\geqslant 0,\ x\in X. \tag{4.1}$$

The following theorem gives the relation between $S(t)$ and $T(t)$.

THEOREM 4.1. *The nonlinear operator $S(t)$ is Fréchet differentiable at $\bar{x}$ and*

$$(DS(t))(\bar{x}) = T(t)$$

for every $t\geqslant 0$.

PROOF. Assume that $\bar{x}=0$. This can always be achieved by setting $\tilde{u}=u-\bar{x}$, $\tilde{F}(\tilde{u})=F(\bar{x}+\tilde{u})-F(\bar{x})$. Let $x\in X$ and $u(t)=S(t)x$. Then $u(t)$ is a solution of the integral equation

$$u(t) = T_0(t)x + \int_0^t T_0^{\odot *}(t-s)Bu(s)ds + \int_0^t T_0^{\odot *}(t-s)\{F(u(s))-Bu(s)\}ds$$

and from Proposition 2.5 we get that

$$u(t) = T(t)x + \int_0^t T^{\odot *}(t-s)\{F(u(s)) - Bu(s)\}ds,$$

hence

$$S(t)x-T(t)x = \int_0^t T^{\odot *}(t-s)\{F(S(s)x) - BS(s)x\}ds.$$

With this observation the proof becomes straightforward. □

It is not difficult to give a proof of this theorem without taking recourse to Proposition 2.5, but instead using Gronwall's lemma.

Let $\omega_0(A)$ denote the type of the semigroup $T(t)$: recall that A is the generator of $T(t)$. We are now ready to prove the following version of one part of the principle of linearized stability which is due to DESCH and SCHAPPACHER [8].

THEOREM 4.2. *Let $\omega_0(A)<0$. For every ϵ with $0\leqslant\epsilon<-\omega_0(A)$ there exists a $\delta>0$ such that $\|x\|\leqslant\delta$ implies that*

$$e^{\epsilon t}S(t)x\rightarrow 0, \quad t\rightarrow\infty.$$

PROOF. Let $\epsilon\in[0,-\omega_0(A))$ and choose $\gamma\in(\epsilon,-\omega_0(A))$. There is a constant $M_\gamma\geq 1$ such that

$$\|T(t)\|\leq M_\gamma e^{-\gamma t},\quad t\geq 0.$$

Choose t_0 so large that $M_\gamma e^{-(\gamma-\epsilon)t_0}\leq 1/4$, and choose $\delta\in(0,1]$ so small that

$$\|S(t_0)x - T(t_0)x\|\leq\frac{1}{4}e^{-\epsilon t_0}\|x\|\leq\delta.$$

Then, if $\|x\|\leq\delta$,

$$e^{\epsilon t_0}\|S(t_0)x\|\leq e^{\epsilon t_0}\|S(t_0)x-T(t_0)x\| + e^{\epsilon t_0}\|T(t_0)x\|$$
$$\leq\frac{1}{4}\|x\| + \frac{1}{4}\|x\| = \frac{1}{2}\|x\|.$$

Thus, for $k\geq 1$,

$$e^{k\epsilon t_0}\|S(kt_0)x\|\leq(\tfrac{1}{2})^k\|x\|,\quad \|x\|\leq\delta.$$

Every $t\geq 0$ can be written as $t=kt_0+\tau$ for some integer k and some $\tau\in[0,t_0)$. Then

$$e^{\epsilon t}\|S(t)x\| = e^{\epsilon\tau}e^{k\epsilon t_0}\|S(kt_0)S(\tau)x\|\leq$$
$$e^{\epsilon\tau}(\tfrac{1}{2})^k\|S(\tau)x\|\leq e^{\epsilon t_0}(\tfrac{1}{2})^{t/t_0-1}\cdot\delta,\quad t\geq 0$$

if $\|x\|\leq M^{-1}e^{-(\omega+ML)t_0}\delta$. □

Desch and Schappacher also proved an instability result in [8], and it is easy to show that this result carries over to our case immediately, so we omit the proof.

THEOREM 4.3. *Suppose that* $\omega_0(A)>0$, *and assume that* $X=X_1\oplus X_2$, *where* X_1 *is finite-dimensional and where both* X_1 *and* X_2 *are invariant under* $T(t)$. *Let* $T_i(t)$ *denote the restriction of* $T(t)$ *to* X_i *and let* A_i *be the corresponding infinitesimal generator. If*

$$\omega_0(A_2)<\min\{\mathrm{Re}\lambda:\lambda\in\sigma(A_1)\}$$

then there exists a constant $\epsilon>0$ *and a sequence* $\{x_n\}_{n\geq 1}\subset X$ *converging to* 0 *and a sequence* $\{t_n\}_{n\geq 1}\subset\mathbb{R}$, *converging to* ∞ *such that* $\|S(t_n)x_n\|\geq\epsilon$.

5. PERTURBATIONS WITH FINITE DIMENSIONAL RANGE AND EXTENSIONS TO THE "BIG" SPACE

In the linear case we can, for an arbitrary C_0-semigroup $T(t)$, define the extension $T^{\odot *}(t)$ to the "big" space $X^{\odot *}$ by taking second semigroup adjoints. This technique breaks down in the nonlinear case. In this section we introduce a more direct alternative technique which does work in the nonlinear case, but which is bound to special perturbations with finite dimensional range. Still other techniques are conceivable, but we will not go into these here.

For ease of formulation we will restrict ourselves to the one-dimensional case. More precisely we take $F:X\rightarrow X^{\odot *}$ to be of the special form

$$F(x) = G(\langle x,r^*\rangle)r^{\odot *} \tag{5.1}$$

for some $r^*\in X^*$, $r^{\odot *}\in X^{\odot *}$ and $G:\mathbb{R}\rightarrow\mathbb{R}$ a globally Lipschitz continuous function. As in [2, section 5] we note that

$$Q(t) = \langle\int_0^t T_0^{\odot *}(\tau)r^{\odot *}d\tau,r^*\rangle \tag{5.2}$$

is locally Lipschitz continuous and hence can be written as

$$Q(t) = \int_0^t K(\tau)d\tau \tag{5.3}$$

for some $K \in L_\infty^{loc}$. For given $x \in X$ we define

$$h(t) = \langle T_0(t)x, r^* \rangle, \quad y(t) = \langle S(t)x, r^* \rangle. \tag{5.4}$$

Lemma 5.1 of [2] then implies that

$$y = h + K * G(y), \tag{5.5}$$

where $*$ denotes the convolution product. In other words, y is the solution of a nonlinear renewal (i.e. Volterra convolution) equation. Conversely the formula

$$S(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(t-\tau) r^{\odot *} G(y(\tau)) d\tau \tag{5.6}$$

enables us to reconstruct the semigroup $S(t)x$ from the solution y of (5.5), the unperturbed semigroup $T_0^{\odot *}(t)$ and $r^{\odot *}$. This observation suggests that we may define

$$S^{\odot *}(t)x^{\odot *} = T_0^{\odot *}(t)x^{\odot *} + \int_0^t T_0^{\odot *}(t-\tau) r^{\odot *} G(y(\tau)) d\tau \tag{5.7}$$

provided (i) we can give a proper definition of h for initial data in $X^{\odot *}$, (ii) we are able to solve (5.5) for such h, (iii) we can give a meaning to the integral in (5.7).

The definition of h proceeds exactly as the definition of K: for given $x^{\odot *} \in X^{\odot *}$ there exists $h \in L_\infty^{loc}$ such that

$$\langle \int_0^t T_0^{\odot *}(\tau) x^{\odot *} d\tau, r^* \rangle = \int_0^t h(\tau) d\tau, \tag{5.8}$$

for all $t \geq 0$.

THEOREM 5.1. *For given $x^{\odot *} \in X^{\odot *}$ define $S^{\odot *}(t)x^{\odot *}$ by (5.7) where y is the unique L_∞^{loc} solution of (5.5) corresponding to $h \in L_\infty^{loc}$ defined by (5.8). Then $S^{\odot *}(t)$ is a semigroup whose restriction to X is exactly $S(t)$. Moreover, $t \mapsto S^{\odot *}(t)x^{\odot *}$ is weakly * continuous.*

PROOF. We start by proving the last assertion. The Lebesgue integral

$$\langle \int_0^t T_0^{\odot *}(t-\tau) r^{\odot *} G(y(\tau)) d\tau, x^{\odot} \rangle = \int_0^t \langle r^{\odot *}, T_0^{\odot}(t-\tau) x^{\odot} \rangle G(y(\tau)) d\tau$$

is the convolution of a continuous function and an L_∞-function and is therefore a continuous function of t. Since $t \mapsto T_0^{\odot *}(t)x^{\odot *}$ is weakly * continuous formula (5.7) then implies that $t \mapsto S^{\odot *}(t)x^{\odot *}$ has the same property. Since

$$\begin{aligned} S^{\odot *}(t+s)x^{\odot *} &= T_0^{\odot *}(t+s)x^{\odot *} + \int_0^{t+s} T_0^{\odot *}(t+s-\tau) r^{\odot *} G(y(\tau)) d\tau \\ &= T_0^{\odot *}(t+s)x^{\odot *} + \int_0^s T_0^{\odot *}(t+s-\tau) r^{\odot *} G(y(\tau)) d\tau + \int_0^t T_0^{\odot *}(t-\tau) r^{\odot *} G(y(\tau+s)) d\tau \\ &= T_0^{\odot *}(t) S^{\odot *}(s) x^{\odot *} + \int_0^t T_0^{\odot *}(t-\tau) r^{\odot *} G(y(\tau+s)) d\tau, \end{aligned}$$

the identity $S^{\odot *}(t+s)x^{\odot *} = S^{\odot *}(t) S^{\odot *}(s) x^{\odot *}$ follows *provided* $y(s+\cdot)$ is the solution of the renewal equation (5.5) with forcing function h_s such that

$$\langle \int_0^t T_0^{\odot *}(\tau) S^{\odot *}(s) x^{\odot *} d\tau, r^* \rangle = \int_0^t h_s(\tau) d\tau.$$

On the other hand the fact that y satisfies (5.5) with the original forcing function h implies that

$$y(t+s) = h(t+s) + \int_0^{t+s} K(t+s-\tau)G(y(\tau))d\tau$$

$$= h(t+s) + \int_0^s K(t+s-\tau)G(y(\tau))d\tau + \int_0^t K(t-\tau)G(y(\tau+s))d\tau,$$

or, in other words, that $y(s+\cdot)$ is the solution of the Volterra equation (5.5) with forcing function $h(t+s) + \int_0^s K(t+s-\tau)G(y(\tau))d\tau$. So it only remains to be shown that

$$\int_0^t h(\tau+s)d\tau + \int_0^t \int_0^s K(\tau+s-\sigma)G(y(\sigma))d\sigma d\tau = <\int_0^t T_0^{\odot *}(\tau)S^{\odot *}(s)x^{\odot *}d\tau, r^*>.$$

Now observe that

$$<\int_0^t T_0^{\odot *}(\tau)T_0^{\odot *}(s)x^{\odot *}d\tau, r^*> = <\int_s^{t+s} T_0^{\odot *}(\sigma)x^{\odot *}d\sigma, r^*> = \int_s^{t+s} h(\sigma)d\sigma = \int_0^t h(\tau+s)d\tau,$$

and that

$$<\int_0^t T_0^{\odot *}(\tau)\int_0^s T_0^{\odot *}(s-\sigma)r^{\odot *}G(y(\sigma))d\sigma d\tau, r^*> = \int_0^s <\int_0^t T_0^{\odot *}(\tau+s-\sigma)r^{\odot *}d\tau, r^*> G(y(\sigma))d\sigma$$

$$= \int_0^s \int_{s-\sigma}^{t+s-\sigma} K(\tau)d\tau G(y(\sigma))d\sigma = \int_0^t \int_0^s K(\tau+s-\sigma)G(y(\sigma))d\sigma.$$

When combined with (5.7) these two observations yield the required identity.

Finally, that $S(t)$ is the restriction of $S^{\odot *}(t)$ to X is precisely the content of formula (5.6) □

REMARKS i) Many delay equations as well as many age dependent population equations are described by perturbations of the form (5.1).

ii) In both of these applications $\int_0^t T_0^{\odot *}(t-\tau)r^{\odot *}h(\tau)d\tau \in X$ for any L_∞-function h and, moreover, $T_0^{\odot *}(t)x^{\odot *} \to 0$ for $t \to \infty$. As a consequence all of the interesting dynamics occurs in X.

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