Department of Mathematics and Statistics
Real Analysis I
Exercise 6
2.5.2014

1. (a) Let $G \subset \mathbb{R}^n$ be an open set. Prove that every $x \in G$ is a density point of G.
   (b) Show by an example that a point $x \in \mathbb{R}^n \setminus G$ can be a density point of an open set $G \subset \mathbb{R}^n$.
   (c) Construct a set $A \subset \mathbb{R}$ that is not open although every point $x \in A$ is a density point of A.

2. Let $\mathbb{Q} = \{q_k \colon k \in \mathbb{N}\}$ be the set of all rational numbers. Define
$$f(x) = \sum_{\substack{k\in\mathbb{N}\\ q_k \le x}} 2^{-k}, \quad x \in \mathbb{R}.$$
   Prove:
   (a) $f \colon \mathbb{R} \to (0,1)$ is strictly increasing.
   (b) f is continuous at $x \iff x \in \mathbb{R} \setminus \mathbb{Q}$.

3. Let $f \colon [a,b] \to \mathbb{R}$ and $g \colon [a,b] \to \mathbb{R}$ be functions of bounded variation. Prove that
$$V_{fg}(a,b) \le M_f V_g(a,b) + M_g V_f(a,b),$$
   where $M_f = \sup\{|f(x)| \colon x \in [a,b]\}$ and $M_g = \sup\{|g(x)| \colon x \in [a,b]\}$.

4. (a) Verify that every function of bounded variation is bounded.
   (b) Let $f \colon [a,b] \to \mathbb{R}$ be absolutely continuous. Prove that f is continuous and of bounded variation.

5. Let $f \colon [a,b] \to \mathbb{R}$ be absolutely continuous and let $E \subset [a,b]$ be a set of measure zero ($m(E) = 0$). Prove that also the image fE is of measure zero.

6. Is the function $f \colon [0,1] \to \mathbb{R}$,
$$f(x) = \begin{cases} x^3 \cos \frac{\pi}{x}, & 0 < x \le 1, \\ 0, & x = 0, \end{cases}$$
   absolutely continuous?