Department of Mathematics and Statistics
Real Analysis I
Exercise 5
24-25.4.2014

1. Let (X, Γ, μ) be a measure space, $f \in L^p(X)$, and $t > 0$. Prove the so-called *Chebyshev's inequality*
$$\mu\big(\{x \in X \colon |f(x)| > t\}\big) \le \left(\frac{\|f\|_p}{t}\right)^p .$$

2. Let (X, Γ, μ) be a measure space, $f \colon X \to [0, +\infty]$ a measurable function, and $0 < p < \infty$. Prove that
$$\int_X f^p d\mu = p \int_0^\infty t^{p-1} \mu\big(\{x \in X \colon f(x) > t\}\big) dt \,.$$

3. We say that a measurable function $f \colon \mathbb{R}^n \to \dot{\mathbb{R}}$ belongs to the "weak L^1-space" weak-$L^1(\mathbb{R}^n)$ if there exists a constant $c = c_f < \infty$ such that
$$m\big(\{x \in \mathbb{R}^n \colon |f(x)| > t\}\big) \le \frac{c}{t} \quad \forall t > 0.$$
   (a) Verify that $L^1(\mathbb{R}^n) \subset$ weak-$L^1(\mathbb{R}^n)$.
   (b) Show by a counterexample that weak-$L^1(\mathbb{R}) \not\subset L^1(\mathbb{R})$.

4. The "centered" maximal function $\tilde{M} f \colon \mathbb{R}^n \to [0, \infty]$ of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is defined by
$$\tilde{M} f(x) = \sup_{r>0} \frac{1}{m\big(B(x,r)\big)} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^n.$$
   (a) Prove that $\tilde{M} f(x) \le M f(x) \le 2^n \tilde{M} f(x)$ $\forall x \in \mathbb{R}^n$ for functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.
   (b) Let $f = \chi_{[a,b]}$, $a < b$. Determine Mf and $\tilde{M} f$, and compare these with the case (a).

5. Let $f \in L^1(\mathbb{R}^n)$. Prove that $M f(x) < +\infty$ for a.e. $x \in \mathbb{R}^n$.

6. Let $f \colon \mathbb{R}^n \to \dot{\mathbb{R}}$ be measurable. Prove that
$$m\big(\{x \in \mathbb{R}^n \colon M f(x) > t\} \le \frac{2 \cdot 5^n}{t} \int_{\{x \colon |f(x)| > t/2\}} |f(y)| dy$$
for all $t > 0$.