Department of Mathematics and Statistics
Real Analysis I
Exercise 4
10-11.4.2014

1. Let $1 \le p < \infty$ and $f, f_k \in L^p(X),\ k \in \mathbb{N}$. Suppose that
$$f_k \to f \text{ a.e.} \quad \text{and} \quad \|f_k\|_p \to \|f\|_p.$$
Prove that $\|f_k - f\|_p \to 0$, in other words, $f_k \to f$ in $L^p(X)$.

2. Give an example of functions $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ such that the function
$$y \mapsto f(x-y)g(y)$$
is not integrable for all $x \in \mathbb{R}$.

3. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuous and $g_\varepsilon = m\big(B(0,\varepsilon)\big)^{-1}\chi_{B(0,\varepsilon)}$. Prove that $g_\varepsilon * f(x) \to f(x)$ as $\varepsilon \to 0+$.

4. Suppose that $f \in L^p(\mathbb{R}^n),\ 1 \le p \le \infty$, and $g \in L^1(\mathbb{R}^n)$. Prove that $f * g(x)$ exists for a.e. $x \in \mathbb{R}^n,\ f * g \in L^p(\mathbb{R}^n)$, and
$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

5. Let $g \ge 0$ be a measurable function in $\mathbb{R}^n$. Suppose that there exists a constant C such that $\|f * g\|_p \le C\|f\|_p$ for every non-negative $f \in L^p(\mathbb{R}^n)$. Prove that $C \ge \|g\|_1$.

6. Prove the following covering theorem: Let $\{B(x_i, r_i)\}, i \in I$, be a finite collection of balls in a metric space (X, d). Then there exists $J \subset I$ such that the balls $B(x_i, r_i), i \in J$, are mutually disjoint and
$$\bigcup_{i \in I} B(x_i, r_i) \subset \bigcup_{i \in J} B(x_i, 3r_i).$$