Department of Mathematics and Statistics
Real Analysis I
Exercise 3
3-4.4.2014

1. Let $\mu(X) < \infty$. Prove that
$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$$
for all $f \in L^\infty(\mu)$. Does this hold without the assumption $\mu(X) < \infty$? Why?

2. Suppose that $f \in L^r$ for some $r \in]1, \infty[$. Prove that
$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

3. Let $f \in L^p(X)$, with $p > 1$. Prove that there exists $g \in L^q(X)$, $q = \frac{p}{p-1}$, such that $\|g\|_q = 1$ and
$$\int_X fg\, d\mu = \|f\|_p.$$
Conclusion?

4. Let (X, Γ, μ) be a complete measure space such that $\mu(A) \geq 1$ whenever $A \in \Gamma$ and $\mu(A) > 0$. Suppose that $1 \leq p < q \leq \infty$. Prove that $L^p \subset L^q$ and
$$\|f\|_\infty \leq \|f\|_q \leq \|f\|_p \leq \|f\|_1.$$

5. Let (X, Γ, μ) be a complete measure space and $1 \leq p, q < \infty$.
   (a) Suppose that $f_i \in L^p$, $i \in \mathbb{N}$, $\|f_i - f\|_p \to 0$, and $f_i \to g$ a.e. Prove that $f = g$ a.e.
   (b) Suppose that $f_i \in L^p \cap L^q$, $i \in \mathbb{N}$, $\|f_i - f\|_p \to 0$, and $\|f_i - g\|_q \to 0$ a.e. Prove that $f = g$ a.e.

6. Prove that one can not, in general, choose the set F in Egorov's theorem so that $\mu(X \setminus F) = 0$.