Department of Mathematics and Statistics
Real Analysis I
Exercise 2
27-28.3.2014

1. Let (X, Γ, μ) be a complete measure space and let $f \in L^\infty(X)$. Prove that
$$\|f\|_\infty = \inf\big\{\sup\{|f(x)| \colon x \in X \setminus N\} \colon N \in \Gamma, \mu(N) = 0\big\}.$$

2. Let $\mu(X) < \infty$ and $1 \leq q < p < \infty$.
   (a) Prove, *without using Hölder's inequality*, that $L^p(\mu) \subset L^q(\mu)$.
   (b) Prove (by using Hölder's inequality) that
$$\|f\|_q \leq \|f\|_p \left(\mu(X)\right)^{(p-q)/pq},$$
   for $f \in L^p(\mu)$.

3. Let (f_j) be a sequence of nonnegative Lebesgue-measurable functions $f_j \colon \mathbb{R} \to \mathbb{R}$ such that
$$\int_{\mathbb{R}} f_j \, dm \to 0.$$
Is it true that necessarily $f_j(x) \to 0$ for a.e. $x \in \mathbb{R}$? [Why??]

4. Let $f \colon B^n(0,1) \to \dot{\mathbb{R}}$, $f(x) = \log|x|$. Determine all the values of p for which $f \in L^p(B^n(0,1))$?

5. Suppose that $f \in L^p$ and $g \in L^q$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for some $r \geq 1$. Prove that $fg \in L^r$ ja $\|fg\|_r \leq \|f\|_p \|g\|_q$.

6. Suppose that $f, f_k \in L^1(\mu)$, $f_k \geq 0$, $k \in \mathbb{N}$, and $\liminf_{k\to\infty} f_k(x) \geq f(x)$ for μ-a.e. $x \in X$. Suppose furthermore that
$$\int_X f_k d\mu = \int_X f d\mu = 1$$
for all $k \in \mathbb{N}$. Prove that
$$\int_X |f_k - f| d\mu \to 0.$$