Department of Mathematics and Statistics
Real Analysis I
Exercise 1
20-21.3.2014

1. Let (X, Γ, μ) be a measure space and $E \in \Gamma$. Prove that
   (a) $\Gamma_E = \{A \in \Gamma \colon A \subset E\}$ is a σ-algebra in E and
   (b) $\mu | \Gamma_E$ is a measure in E.

2. Let X be a nonempty set. Define $\mu^* \colon \mathcal{P}(X) \to \{0, 1\}$ by setting $\mu^*(\emptyset) = 0$ and $\mu^*(A) = 1$ if $A \neq \emptyset$. Prove that μ^* is an outer measure. What are the μ^*-measurable sets?

3. Let $I = [0, 1] \times [0, 1]$ and $f(x, y) = (x - y)/(x + y)^3$ when $(x, y) \in I \setminus \{(0, 0)\}$ and $f(0, 0) = 0$. Compute the integrals
$$\int_0^1 \left(\int_0^1 f(x, y)\, dy \right) dx \quad \text{ja} \quad \int_0^1 \left(\int_0^1 f(x, y)\, dx \right) dy.$$
   Is f integrable in I?

4. For each $A \subset \mathbb{R}^n$ define
$$\tilde{A} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \colon x - y \in A\}.$$
   Prove that $m_{2n}(\tilde{A}) = 0$ if $m_n(A) = 0$.

5. Prove that $B \subset \mathbb{R}$ is a Borel set if and only if it is a Borel set also as a subset of the plane. More precisely,
$$\operatorname{Bor} \mathbb{R} = \{B \subset \mathbb{R} \colon \{(x, 0) \colon x \in B\} \in \operatorname{Bor} \mathbb{R}^2\}.$$
   Is there a similar relation between σ-algebras $\operatorname{Leb} \mathbb{R}$ and $\operatorname{Leb} \mathbb{R}^2$?

6. Find a sequence $p = (p_1, p_2, \ldots)$ so that the corresponding Cantor set $C(p)$ has measure (1-dimensional Lebesgue measure) $\frac{1}{3}$.