Introduction to Finite Element Method
Exercises

---

1 Consider the following 1D second order differential equation

$$-\frac{d^2}{dx^2}u(x) + u(x) = f(x), \quad x \in [a, b],$$

with the Dirichlet and the Neumann boundary conditions

$$u(a) = g^D(a) \quad \text{and} \quad u(b) = g^D(b),$$

and

$$\frac{du}{dx}(a) = g^N(a) \quad \text{and} \quad \frac{du}{dx}(b) = g^N(b),$$

respectively. Here f, g^D and g^N are known functions.

Derive the weak formulations of these boundary value problems and the expressions for the elements of the system matrices (matrix $\boldsymbol{A}$) and source vectors (vector $\boldsymbol{b}$).

Derive also analytical solutions for the problems with zero boundary data ($g^D(a) = 0 = g^D(b)$ and $g^N(a) = 0 = g^N(b)$) with $a = 0$ and $b = \pi$. For the Neumann problem the source function $f(x) = -2\cos(x)$ and for the Dirichlet problem $f(x) = 2\sin(x)$.

2 Consider the same 1D differential equation as above. Write a Matlab code which computes the system matrix $\boldsymbol{A}$ and source vector $\boldsymbol{b}$ with the homogeneous Neumann boundary condition

$$\frac{du}{dx}(a) = 0 \quad \text{and} \quad \frac{du}{dx}(b) = 0,$$

using piece-wise linear basis and testing functions. Syntax of the program (Matlab function) is

```
[A,bv] = exercise2(a,b,K,P,fun)
```

where `a` and `b` are the end points of an interval, `K` is the number of elements of the finite element approximation, `P` is the number of Gauss-Legendre integration points (e.g., 2), `fun` is a handle to function f, `A` is a $(K + 1) \times (K + 1)$ matrix (system matrix) and `bv` is a $(K + 1) \times 1$ vector (source vector).

3 Modify your program so that it can solve the Dirichlet problem with homogeneous boundary conditions

$$u(a) = 0 \quad \text{and} \quad u(b) = 0.$$

Syntax of the program (Matlab function) is

```
[A,bv] = exercise3(a,b,K,P,fun)
```

where the parameters are the same as in exercise 2.

4 Use programs `exercise2.m` and `exercise3.m` to solve the Neumann problem with $g^N(a) = 0 = g^N(b)$, $a = 0, b = \pi$ and $f(x) = -2\cos(x)$, and the Dirichlet problem with $g^D(a) = 0 = g^D(b)$, $a = 0, b = \pi$ and $f(x) = 2\sin(x)$. Compare numerical solutions with analytical ones and compute the relative L^2 error estimates of the approximations u^h for both problems. The L^2 error estimate is

$$\frac{\|u_{\text{ana}} - u^h\|_2}{\|u_{\text{ana}}\|_2} := \frac{\sqrt{\int\limits_a^b |u_{\text{ana}}(x) - u^h(x)|^2 \, dx}}{\sqrt{\int\limits_a^b |u_{\text{ana}}(x)|^2 \, dx}},$$

where u_{ana} is the analytical solution.

Study convergences of the FEM solutions for the Neumann and Dirichlet problems by increasing the number of elements (parameter K). What are the orders of the convergences of the solutions versus the element size?

Hint 1: Program `gausslegendre.m` can be used to obtain the points and weights of Gauss-Legendre method on 1D reference element (a line segment $[0, 1]$).

Hint 2: Program `plot_approx_linear.m` can be used to plot a piece-wise linear approximation in 1D.

Hint 3: Program `compute_error_linear.m` can be used to compute a L^2 error of piece-wise linear approximation in 1D.

Hint 4: Assume that we would like to evaluate function $f(x) = x^2$ at 10 points on interval $[0, 1]$. This can be done using a function handle as follows:

```
fun = @(x)x.^2;
x = linspace(0,1,10);
f = fun(x);
```

and f has values `0 0.0123 0.0494 0.1111 0.1975 0.3086 0.4444 0.6049 0.7901 1.0000`.
