Department of Mathematics and Statistics
Measure and Integral
Exercise 6
24-28.2.2014

1. Define functions $f_j\colon [0,1] \to [0,1]$ as follows: $f_1 = \chi_{[0,1/2]}$, $f_2 = \chi_{[1/2,1]}$, $f_3 = \chi_{[0,1/4]}$, $f_4 = \chi_{[1/4,1/2]}$, $f_5 = \chi_{[1/2,3/4]}$, $f_6 = \chi_{[3/4,1]}$, $f_7 = \chi_{[0;1/8]}, \dots$. Find the functions $\limsup_{j\to\infty} f_j$ and $\liminf_{j\to\infty} f_j$.

2. Find the limit
$$\lim_{n\to\infty} \int_0^1 \frac{nx^s}{1+nx}\,dx,$$
when $0 < s < 1$.

3. Let $E_j \subset \mathbb{R}^n$, $j \in \mathbb{N}$, be a sequence of measurable disjoint sets and let $f\colon \mathbb{R}^n \to [0,+\infty]$ be a measurable function. Prove that
$$\int_{\cup_j E_j} f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

4. ["Decreasing monotone convergence theorem"]
Let $f_j\colon E \to \mathbb{R}$ be measurable functions such that $f_1 \geq f_2 \geq \cdots \geq 0$. Prove: If $\int_E f_1 < \infty$, then
$$\int_E \lim_{j\to\infty} f_j = \lim_{j\to\infty} \int_E f_j.$$

5. Let
$$f_n(x) = \frac{1+e^{-n|x|}}{1+x^2},\ x \in \mathbb{R},\ n \in \mathbb{N}.$$
Find the limit
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

6. Let $(f_j)_{j=1}^{\infty}$ be a sequence of measurable functions $f_j\colon \mathbb{R}^n \to \mathbb{R}$ such that the sum $\sum_{j=1}^{\infty} |f_j|$ is integrable. Prove that
$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f_j = 0.$$