Department of Mathematics and Statistics
Measure and Integral
Exercise 5
17-21.2.2014

1. Prove that the product fg of measurable functions $f, g\colon A \to \dot{\mathbb{R}}$ is measurable.

2. Find $\liminf_{k\to\infty} a_k$ and $\limsup_{k\to\infty} a_k$ when:
   (a)
$$a_k = \frac{1 + 2k\sin\frac{k\pi}{8}}{3+4k};$$
   (b)
$$a_k = (-1)^k \frac{k^2+5k}{2k^2-4k};$$
   (c)
$$a_k = \sum_{j=1}^{k} (-1)^{k+j} 2^{-j}.$$

3. Suppose that a sequence $(a_k)_{k=1}^{\infty}$ has a subsequence $(a_{k_j})_{j=1}^{\infty}$, with a limit $\lim_{j\to\infty} a_{k_j} = a$. Prove that
$$\liminf_{k\to\infty} a_k \le a \le \limsup_{k\to\infty} a_k.$$

4. Let $f_1, f_2, \ldots$ be measurable functions $A \to \mathbb{R},\ A \subset \mathbb{R}^n$. Prove that the set $B = \{x \in A\colon\ \exists \lim_{j\to\infty} f_j(x)\}$ is measurable.

5. Let $A \subset \mathbb{R}^n$ be a measurable set and $f_j\colon A \to \mathbb{R},\ j \in \mathbb{N}$, a sequence of measurable functions.
   (a) Prove that the sets
$$A_j = \{x \in A\colon f_{j+1}(x) > f_j(x)\}$$
   are measurable
   (b) Prove that the set
$$\{x \in A\colon\ (f_k(x))_{k=1}^{\infty} \text{ strictly increasing}\}$$
   is measurable.

6. Let $f\colon A \to \mathbb{R}$ be measurable. Prove that its positive part $f^+\colon A \to \mathbb{R},\ f^+(x) = \max(0, f(x))$, is measurable.