Department of Mathematics and Statistics
Measure and Integral
Exercise 4
10-14.2.2014

1. Prove that the set $A = \{(x, y) \in \mathbb{R}^2 \colon x > 1 \text{ ja } 0 \leq yx^2 < 1\}$ is measurable.

2. Let $A \subset \mathbb{R}^n$ and $f \colon A \to \mathbb{R}$. Let
$$A_r = \{x \in A \colon f(x) > r\},$$
when $r \in \mathbb{R}$. Prove: if $m_n^*(A_0) > 0$, there exists $r > 0$ such that $m_n^*(A_r) > 0$.

3. Let A_j, $j \in J$, be disjoint measurable subsets of $\mathbb{R}^n$ such that $m_n(A_j) > 0$ for all $j \in J$. Prove that J is countable.

4. Prove that there exists disjoint sets $A \subset \mathbb{R}$ and $\mathbb{B} \subset \mathbb{R}$ such that
$$m^*(A \cup B) < m^*(A) + m^*(B).$$

5. Let $A_k \subset [0, 1]$, $k = 1, 2, \ldots$, be measurable. Suppose that
$$m(A_k) > \frac{2^k - 1}{2^k}.$$
for all $k \in \mathbb{N}$. Prove that the intersection $\bigcap_{k=1}^{\infty} A_k$ is non-empty.

6. Prove that the function $f \colon \mathbb{R} \to \mathbb{R}$,
$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}; \\ x^2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$
is measurable.