Department of Mathematics and Statistics
Measure and Integral
Exercise 2
27-31.1.2014

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (t_1x_1, t_2x_2)$, where $t_1, t_2 \in \mathbb{R}$. Prove that
$$m_2^*(fA) = |t_1t_2| m_2^*(A)$$
for all $A \subset \mathbb{R}^2$.

2. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *Lipschitz* if there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$. Prove: If $A \subset \mathbb{R}$ is of zero measure and $f\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, then also the image fA is measurable and $m(fA) = 0$.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $A \subset \mathbb{R}$, $m(A) = 0$. Prove that $m(fA) = 0$. [Consider first the case where A is bounded by using the mean value theorem.]

4. Let $A \subset [2, 4]$. Prove that
$$12\, m_1^*(A) \leq m_1^*\big(\{x^3 \colon x \in A\}\big) \leq 48\, m_1^*(A).$$
Can the constant 12 (resp. 48) be replaced by a bigger (resp. smaller) constant?

5. Prove that a set $E \subset \mathbb{R}^n$ is measurable if and only if
$$m^*(S \cup U) = m^*(S) + m^*(U)$$
for all $S \subset E$ and $U \subset \mathbb{R}^n \setminus E$.

6. Prove that a set $E \subset \mathbb{R}^n$ is measurable if and only if
$$m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$$
for every open n-interval I. [You can use the fact that $m^*(I) = \ell(I)$ for an n-interval I.]