Department of Mathematics and Statistics
Measure and Integral
Exercise 1
20-24.1.2014

1. (a) Find $\inf E$ and $\sup E$ when $E = \{\frac{1}{\log x} \in \mathbb{R} \colon x > 1\}$.
   (b) Suppose that $\emptyset \neq B \subset A \subset \mathbb{R}$. Prove that
   $$\inf A \leq \inf B \leq \sup B \leq \sup A.$$
   (c) Let $\emptyset \neq A \subset \mathbb{R}$ and $-2A = \{-2x \colon x \in A\}$. Prove that
   $$\inf(-2A) = -2 \sup A.$$

2. Let $V_1, \ldots, V_k \subset \mathbb{R}^n$ be open and $F_1, \ldots, F_k \subset \mathbb{R}^n$ closed subsets of $\mathbb{R}^n$. Prove that $\cap_{j=1}^k V_j$ is open and $\cup_{j=1}^k F_k$ is closed. Give examples of the following phenomena: (a) $V_j \subset \mathbb{R}$ is open for every $j \in \mathbb{N}$ but the intersection $\cap_{j=1}^\infty V_j$ is not open; (b) $F_j \subset \mathbb{R}$ is closed for every $j \in \mathbb{N}$ but the union $\cup_{j=1}^\infty F_j$ is not closed.

3. Let I be an uncountable set and $a_i > 0$ for every $i \in I$. Prove that
   $$\sum_{i \in I} a_i := \sup_{J \subset I \text{ finite}} \sum_{j \in J} a_j = +\infty.$$

4. Let
   $$\mathcal{B} = \{B^n(x,r) \subset \mathbb{R}^n \colon x = (x_1, \ldots, x_n) \in \mathbb{Q}^n, r \in \mathbb{Q}, r > 0\}.$$
   Prove that $\mathcal{B}$ is countable. [In other words, $\mathcal{B}$ is the family of all open balls $B^n(x,r) \subset \mathbb{R}^n$ whose radii r are positive rational numbers and the coordinates of the centres x are rational numbers.]

5. Let $A \subset \mathbb{R}^2$ be countable. Prove by using the definition of an outer measure that $m_2^*(A) = 0$.

6. Let $f \colon [0,1] \to \mathbb{R}$ be continuous and $G_f = \{(x, f(x)) \in \mathbb{R}^2 \colon x \in [0,1]\}$ its graph. Prove that $m_2^*(G_f) = 0$.