SOME BASIC FACTS FROM MARTINGALE THEORY

DARIO GASBARRA

1. Conditional Expectation and Martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 1. *Conditional expectation: Let X be a random variable, (which is $\mathcal{F}$-measurable) and a sub σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, $E_P(X|\mathcal{G})$ is a $\mathcal{G}$-measurable random variable such that for all $B \in \mathcal{G}$*

$$E_P(\mathbf{1}_B X) = E_P(\mathbf{1}_B E_P(X|\mathcal{G}))$$

Properties: i) $E_P(E_P(X|\mathcal{G})) = E_P(X)$,
ii) if Y is $\mathcal{G}$-measurable $E_P(XY|\mathcal{G}) = Y E_P(X|\mathcal{G})$.
iii) if $Y \perp\!\!\!\perp \mathcal{G}$, $E_P(Y|\mathcal{G}) = E_P(Y)$.

iv) If $E_P(X^2) < \infty$, the random variable $E_P(X|\mathcal{G})$ is the orthogonal projection of the r.v. X to the subspace $L^2(\Omega, \mathcal{G}, P) \subset L^2(\Omega, \mathcal{F}, P)$:

$$E((X - E_P(X|\mathcal{G}))^2) = \min_{Y \in L^2(\Omega, \mathcal{G}, P)} E((X - Y)^2) \ .$$

v) the conditional expectation is linear:
$E_P(aX + bY|\mathcal{G})(\omega) = aE_P(X|\mathcal{G})(\omega) + bE_P(Y|\mathcal{G})(\omega)$
vi) The conditional expectaion is linear is non-negative, if $X(\omega) \geq 0$ P a.s. , then $E(X|\mathcal{G})(\omega) \geq 0$ P a.s.
Let Q a probability measure which dominates P ($P \ll Q$) on a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, which means that $Q(A) = 0 \implies P(A) = 0$ for all $A \in \mathcal{G}$. The Radon-Nikodym derivative of P w.r.t Q is a $\mathcal{G}$-measurable random variable

$$Z^{\mathcal{G}}(\omega) = Z^{\mathcal{G}}(P, Q)(\omega) = \frac{dP|\mathcal{G}}{dQ|\mathcal{G}}(\omega) \geq 0$$

This means that $P(d\omega) = Z(P, Q)(\omega)Q(d\omega)$ on $\mathcal{G}$, and if X is a $\mathcal{G}$-measurable random variable we change the measure to represent the expectation w.r.t. P as an expectation w.r.t. Q:

$$E_P(X) = E_Q(XZ(P, Q))$$

We have that $0 \leq Z^{\mathcal{G}}(P, Q) \in L^1(\Omega, \mathcal{G}, Q)$, ja $E_Q(Z(P, Q)) = 1$.
In statistics $Z(P, Q)$ is called likelihood ratio.
Note that if $\mathcal{A} \subseteq \mathcal{G}$ and $P \ll Q$ on $\mathcal{G}$, then trivially $P \ll Q$ on $\mathcal{A}$, and

$$Z^{\mathcal{A}}(P, Q) = E_Q(Z^{\mathcal{G}}(P, Q)|\mathcal{A}).$$

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This is the Q-martingale property for nested σ-algebrae.

We have also a formula to change the measure in the conditional expectation. For $P \ll Q$, $\mathcal{G} \subseteq \mathcal{F}$, and X is $\mathcal{F}$-measurable, *Bayes formula* holds:

$$E_P(X|\mathcal{G}) = \frac{E_Q(XZ(P,Q)|\mathcal{G})}{E_Q(Z(P,Q)|\mathcal{G})}$$

Sometimes it is also called abstract Bayes formula. The proof is not difficult, for $B \in \mathcal{G}$, denoting $Z = Z^{\mathcal{F}}(P,Q)$,

$$E_P(X\mathbf{1}_B) = E_Q(ZX\mathbf{1}_B) = E_Q(E_Q(ZX\mathbf{1}_B|\mathcal{G})) = E_Q(E_Q(ZX|\mathcal{G})\mathbf{1_B})$$
$$= E_Q\left(\frac{E_Q(Z|\mathcal{G})}{E_Q(Z|\mathcal{G})}E_Q(ZX|\mathcal{G})\mathbf{1}_B\right) = E_Q\big(Z\frac{E_Q(ZX|\mathcal{G})}{E_Q(Z|\mathcal{G})}\mathbf{1}_B\big) = E_P\big(\frac{E_Q(ZX|\mathcal{G})}{E_Q(Z|\mathcal{G})}\mathbf{1}_B\big)$$

and the result follows from the definition of conditional expectation.

Example 1. *As an exercise we show that the elementary Bayes formula used in statistics follows as a special case:*
Let (X,Y) a random vector with values in $\mathbb{R}^2$, with

$$P(X \in dx, Y \in dy) = \pi(x)p(y|x)dxdy$$

We work directly on the canonical space $\Omega = \mathbb{R}^2$. On the σ-algebra $\mathcal{F} = \sigma(X,Y)$, we take as reference measure a dominating product measure, for example $Q(dx,dy) = \pi(x)dxdy$ (although Q is not a probability measure, Bayes formula works also in this case).
Clearly $P \ll Q$ and $Z(P,Q) = \frac{dP}{dQ}(x,y) = p(y|x)$.
When we condition to the sub-σ-algebra $\mathcal{G} = \sigma(Y)$, our (abstract) Bayes formula says that for any bounded measurable function $f(x)$,

$$E_P(f(X)|\sigma(Y))(\omega) = \frac{E_Q(f(X)Z(P,Q)|\sigma(Y))(\omega)}{E_Q(Z(P,Q)|\sigma(Y))(\omega)} = \frac{\int_{\mathbb{R}} f(x)\pi(x)p(Y(\omega)|x)dx}{\int_{\mathbb{R}} \pi(x)p(Y(\omega)|x)dx}$$

which is the elementary Bayes formula as we use it in statistics.

We introduce now a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$, which is an increasing sequence of σ-algebrae shich that, for all $s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.
(here it does not matter whether the time is discrete or continuous, we can always imbed discrete time in continuous time by taking $\mathcal{F}_t = \mathcal{F}_{\lfloor t \rfloor}$).

Definition 2. *A process M_t is a $(P,\mathbb{F})$-martingale if M_t is $\mathcal{F}_t$ measurable, $M_t \in L^1(P)$, and for $s \leq t$*

$$E_P(M_t|\mathcal{F}_s) = M_s \ .$$

When

$$E_P(M_t|\mathcal{F}_s) \leq M_s \quad , \ s \leq t$$

we say that (M_t) is a $(P,\{\mathcal{F}_t\})$-supermartingale, and if

$$E_P(M_t|\mathcal{F}_s) \geq M_s \quad , \ s \leq t$$

(M_t) is a $(P,\{\mathcal{F}_t\})$-submartingale.

Given all the past, the conditional expectation of a future value of a martingale is the current value.
Note that the martingale property depends on the measure P and on the filtration $\{\mathcal{F}_t\}$.
Given two measures P and Q defined on $(\Omega, \mathcal{F})$ we consider at each time t the restriction of the measures to the current information σ-algebra $\mathcal{F}_t$, $P_t = P|_{\mathcal{F}_t}$, $Q_t = Q|_{\mathcal{F}_t}$.
If $P_t \ll Q_t$ on $\mathcal{F}_t$, we define

$$Z_t(P,Q) = \frac{dP_t}{dQ_t}.$$

From the definition it follows that $Z_t \in L^1(Q, \mathcal{F}_t)$ and $Z_t(\omega) \geq 0$.
We show that Z_t is a $(Q, \mathbb{F})$ martingale: for $s \leq t$ if $B \in \mathcal{F}_s$ also $B \in \mathcal{F}_t$ and we have

$$P(B) = E_P(\mathbf{1}_B) = E_Q(Z_s \mathbf{1}_B) = E_Q(Z_t \mathbf{1}_B)$$

which means that $Z_s = E_Q(Z_t|\mathcal{F}_s)$.

Example 2. *On a probability space $(\Omega, \mathcal{F})$ we have a sequence of (real valued) random variables $(X_1, X_2, \dots, X_n, \dots)$, and two probability measures P and Q such that (X_i) are independent and identically distributed under both P and Q. We assume that $P(X_1 \in dx) = f(x) Q(X_1 \in dx)$. Let $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$, $t \in \mathbb{N}$. It follows that*

$$Z_t(P,Q) = \prod_{s \in \mathbb{N}: s \leq t} f_s(X_i) \ .$$

Exercise 1. *Check that $Z(P,Q)$ is a $(Q, \{\mathcal{F}_t\})$-martingale.*

Definition 3. *We say that a process (X_t) is adapted if $X_t \in \mathcal{F}_t$ for all t, and in the discrete-time situation it is predictable if $X_t \in \mathcal{F}_{t-1}$ for all t.*

Theorem 1.1. *(discrete-time Doob-Meyer decomposition).*
If (X_t) is adapted to the filtration $\{\mathcal{F}_t\}$, and $E(|X_t|) < \infty$ for all $t = 0, 1, \dots, T$ then there is an unique decomposition

$$X_t = X_0 + A_t + M_t$$

where A_t is $\{\mathcal{F}_t\}$-predictable and M_t is a $\{\mathcal{F}_t\}$-martingale with $A_0 = 0$ and $M_0 = 0$.
If (X_t) is a supermartingale (respectively submartingale) the process A_t is non-increasing, (respectively non-decrasing submartingale).

Proof

$$\Delta X_t = (\Delta X_t - E_P(\Delta X_t|\mathcal{F}_{t-1})) + E_P(\Delta X_t|\mathcal{F}_{t-1}) = \Delta M_t + \Delta A_t$$

where

$$A_t = \sum_{s=1}^{t} E_P(\Delta X_t|\mathcal{F}_{t-1}), \quad M_t = \sum_{s=1}^{t} \big(\Delta X_t - E_P(\Delta X_t|\mathcal{F}_{t-1})\big)$$

If another Doob decomposition of X existed, $X_t - X_0 = \tilde{A}_t + \tilde{M}_t$ we would have $(M_t - \tilde{M}_t) = (A_t - \tilde{A}_t)$ which means that $(M_t - \tilde{M}_t)$ is a predictable martingale, which is necessarily the constant zero.

Definition 4. *If (Y_t) and (X_t) are sequences we define the stochastic integral of Y with respect to X as the sequence*

$$(Y \cdot X)_t = \sum_{s=1}^{t} Y_s \Delta X_s$$

which is called martingale transform or discrete stochastic integral

Theorem 1.2. *Assume that (Y_t) $\{\mathcal{F}_t\}$-predictable process and (M_t) is a $(P, \{\mathcal{F}_t\})$-martingale. If Y_t is a bounded random variable for all t, or alternatively both Y_t and M_t are square integrable r.v., it follows that $E(|Y_t \Delta M_t|) < \infty$. Under such assumptions, the stochastic integral $(Y \cdot M)_t$ is a martingale.*

Proof: Exercise.

2. Square integrable martingales and predictable bracket

A $(P, \{\mathcal{F}_t\})$-martingale (M_t) is square integrable when $E(M_t^2) < \infty$ for all t.
If M_t, N_t are square integrable martingales then by using Cauchy-Schwartz inequality

$$E(|M_t N_t|) \leq \sqrt{E(M_t^2)} \sqrt{E(N_t^2)} < \infty$$

so that the product $(M_t N_t)$ is in L^1 and it makes sense to consider its Doob-Meyer decomposition:
We have

$$M_t N_t - M_{t-1} N_{t-1} = M_{t-1} \Delta N_t + N_{t-1} \Delta M_t + \Delta M_t \Delta N_t =$$
$$M_{t-1} \Delta N_t + N_{t-1} \Delta M_t + \big(\Delta M_t \Delta N_t - E_P(\Delta M_t \Delta N_t | \mathcal{F}_{t-1})\big) + E_P(\Delta M_t \Delta N_t | \mathcal{F}_{t-1})$$

We introduce the predictable process

$$\langle M, N \rangle_t := \sum_{s=1}^{t} E_P(\Delta M_s \Delta N_s | \mathcal{F}_{s-1})$$

We obtain the Doob-Meyer decomposition

$$M_t N_t = M_0 N_0 + \langle M, N \rangle_t + m_t$$

where dm_t the sum the martingale increments

$$dm_t = M_{t-1} \Delta N_t + N_{t-1} \Delta M_t + \big(\Delta M_t \Delta N_t - E_P(\Delta M_t \Delta N_t | \mathcal{F}_{t-1})\big)$$

where the integrability conditions in the definition of martingale follow from Cauchy-Schwartz inequality since we have assumed M and N are square-integrable.
We denote also

$$[M, N]_t := \sum_{s=1}^{t} \Delta M_s \Delta N_s$$

it follows that the process $([M,N]_t - \langle M,N\rangle_t)$ is a $(P,\{\mathcal{F}_t\})$-martingale.

$[M,N]_t$ is called quadratic covariation or square-bracket process, while $\langle M,N\rangle_t$ is called predictable covariation, or predictable-bracket process.
Since $E((\Delta M_t)_P|\mathcal{F}_{t-1}) \geq 0$, the process $([M,M]_t)$ is a submartingale and therefore $(\langle M,M\rangle_t)$ is non-decreasing. The notations $[M]_t := [M,M]_t$ and $\langle M\rangle_t := \langle M,M\rangle_t$ are also used.
Note $[M,N]_t$ does not depend on the measure P, but the predictable bracket $\langle M,N\rangle_t$ does !

Definition 5. *Two square integrable martingales $(M_t),(N_t)$ are orthogonal if the product (M_tN_t) is a martingale. Equivalent conditions are*
i) $[M,N]_t$ is a martingale,
ii) $\langle M,N\rangle_t = 0$, which means $E_P(\Delta M_t\Delta N_t|\mathcal{F}_{t-1})(\omega) = 0$ P a.s.

Note that this definition extends to the case when M_t is a martingale (not necessarly square integrable) and N_t is a martingale in $L^\infty(P)$ $\forall t$.
Note also that

$\Delta\langle M\rangle_t := E_P((\Delta M_t)^2|\mathcal{F}_{t-1})(\omega) < \infty$ and $E_P(\Delta\rangle M_t\langle) < \infty \Longleftrightarrow E_P(\Delta M_t^2)$

It is possible that $\Delta\langle M\rangle_t < \infty$ $(P=1)$ but $E_P(\Delta M_t^2) = \infty$ (M_t is not integrable). In such case we can still use the notion of predictable covariation and orthogonality of martingales.

3. Orthogonal projections in the space of square integrable martingales

Let M and N two square integrable martingales,
We write

$$N_t = N_0 + (H\cdot M)_t + N_t^\perp = N_0 + \sum_{s=1}^{t} H_s\Delta M_s + N_t^\perp \tag{3.1}$$

where (H_t) is the predictable process

$$H_t = \mathbf{1}\big(\Delta\langle M,M\rangle_t > 0\,\big)\,\frac{\Delta\langle M,N\rangle_t}{\Delta\langle M,M\rangle_t} = \mathbf{1}\big(E_P((\Delta M_t)^2|\mathcal{F}_{t-1}) > 0\,\big)\,\frac{E_P(\Delta M_t\Delta N_t|\mathcal{F}_{t-1})}{E_P((\Delta M_t)^2|\mathcal{F}_{t-1})}$$

and $N_t^\perp$ is a P-martingale orthogonal to M_t.

Note first that since the conditional expectation is a positive operator,

$$E_P\Big(\Delta M_t^2|\mathcal{F}_{t-1})(\omega) \geq 0$$

and therefore

$$E_P\Big(\Delta M_t^2|\mathcal{F}_{t-1})(\omega) = 0$$

if and only if

$$P\Big(\Delta M_t^2 = 0|\mathcal{F}_{t-1})(\omega) = 1$$

otherwise for some $\varepsilon > 0$

$$P\Big(\Delta M_t^2 > \varepsilon|\mathcal{F}_{t-1})(\omega) > \eta > 0$$

which is in contradiction with

$$E_P\Big(\Delta M_t^2|\mathcal{F}_{t-1})(\omega) = 0.$$

This implies

$$E_P\Big(\Delta N_t \Delta M_t|\mathcal{F}_{t-1})(\omega) = 0$$

Note also that $H_t \in L^2(\Omega, \mathcal{F}_{t-1}P)$, since

$$E_P(H_t) = E_P\left(\left\{\frac{E_P(\Delta M_t \Delta N_t|\mathcal{F}_{t-1})}{E_P(\{\Delta M_t\}^2|\mathcal{F}_{t-1})}\right\}^2\right) \le E_P\big(\Delta N_t^2\big) < \infty$$

where we used the Cauchy-Schwartz inequality for the conditional expectation together with the properties of the conditional expectation.

$$\Big\{E_P(\Delta M_t \Delta N_t|\mathcal{F}_{t-1})(\omega)\Big\}^2 \le E_P((\Delta M_t)^2|\mathcal{F}_{t-1})(\omega) E_P((\Delta N_t)^2|\mathcal{F}_{t-1})(\omega)$$

4. Martingale property and change of measure

Theorem 4.1. *Let $Q \ll P$ and let*

$$Z_t(\omega) = Z_t(Q, P) = \frac{dQ_t}{dP_t}(\omega)$$

Then M_t is a $(Q, \{\mathcal{F}_t\})$-martingale if and only if the product $(M_t Z_t)$ is a $(P, \{\mathcal{F}_t\})$-martingale.

Proof for $s \le t$, let $A \in \mathcal{F}_s$.

$$E_Q(1_A(M_t - M_s)) = E_P(1_A Z_t(M_t - M_s)) = E_P(1_A(Z_t M_t - Z_s M_s))$$

where we use the properties of the conditional expectation. By definition of conditional expectation it means that

$$E_Q(M_t|\mathcal{F}_s) = M_s \text{ if and only if } E_P(Z_t M_t|\mathcal{F}_s) = Z_s M_s$$

5. Doob decomposition and change of measure

Suppose that M is a $(P, \mathcal{F}_t)$ martingale with $M_0 = 0$ and $\Delta M_t > -1$.

$$Z_t = \mathcal{E}(M)_t := \prod_{s=1}^{t}(1 + \Delta M_t) = \left(1 + \sum_{s=1}^{t} Z_{s-1}\Delta M_s\right) > 0$$

and we define on each $\mathcal{F}_t$ consistently a measure

$$Q_t(d\omega) = Z_t(\omega) P_t(d\omega)$$

If $(Z_t)_{t=0,1,\dots,T}$ is integrable, then (Z_t) is a P-martingale and $Q_t(\Omega) = E_P(Z_t) = Z_0 = 1$ which is a probability measure.

Example 3. *Assume that $\{\xi_t(\omega) : t = 1, \dots, T\}$ are i.i.d. $\mathcal{N}(0,1)$ distributed (univariate gaussian with 0 mean and variance 1).*
For a given $\theta \in \mathbb{R}$ Define

$$M_t(\theta) = \sum_{s=1}^{t} \{\exp(\theta\xi_s - \frac{1}{2}\theta^2) - 1\}$$

This is a martingale with independent increments, and $\Delta M_t > -1$.
Then we set $Z_0(\theta) = 1$ and

$$Z_t(\theta) = \mathcal{E}(M(\theta))_t = 1 + \sum_{s=1}^{t} Z_{s-1}(\theta)\Delta M_s = \prod_{s=1}^{t}\Big(1 + \exp(\theta\xi_s - \frac{1}{2}\theta^2) - 1\Big) =$$

$$\prod_{s=1}^{t} \exp(\theta\xi_s - \frac{1}{2}\theta^2) = \exp\Big(\theta\sum_{s=1}^{t}\xi_s - \frac{1}{2}\theta^2 t\Big)$$

It follows that $Z_t(\theta)$ is integrable, since under P, the r.v. $\big(\sum_{s=1}^{t}\xi_s\big)$ is gaussian $\mathcal{N}(0,t)$. Since integrability is satisfied, $Z_t(\theta)$ is a P- martingale, which defines a probability measure $dQ_t(\theta) = Z_t(\theta)dP_t$ on $\mathcal{F}_t$.
For example for $N_t = \sum_{s=1}^{t}\xi_t$, the martingale decomposition under $Q(\theta)$ is given by

$$\Delta N_t = \big(\Delta N_t - \Delta\langle N, M(\theta)\rangle_t\big) + \Delta\langle N, M(\theta)\rangle_t$$

$$\big\{\xi_t - E_P\big(\xi_t\exp(\theta\xi_t - \frac{1}{2}\theta^2)\big)\big\} + E_P\big(\xi_t\exp(\theta\xi_t - \frac{1}{2}\theta^2)\big) = \big\{\xi_t - \theta\big\} + \theta$$

meaning that $(N_t - \theta t)$ is a $Q(\theta)$-martingale.
Here

$$E_P\big(\xi_t\exp(\theta\xi_t - \frac{1}{2}\theta^2)\big) = \frac{\partial}{\partial\theta}\log E_P\big(\exp(\theta\xi_t)\big) = \theta$$

Assume that M and N are square integrable P martingales, $\Delta M_t \geq -1$ and $Z_t = \mathcal{E}(M)_t, t = 1, \dots, T$ with $Z_T \in L^1(P)$ for all t.
By projecting N on M obtaining the orthogonal martingale decomposition

$$N_t = N_0 + (H \cdot M)_t + N_t^{\perp}$$

What happens to the martingale property of N and M under the new measure ?

Proposition 5.1. *(Girsanov theorem in discrete time) The Doob decomposition of N under Q is given by*

$$N_t = N_0 + \big(H \cdot \langle M, M\rangle\big)_t + \big(H \cdot (M - \langle M, M\rangle)\big)_t + N_t^{\perp}$$

where $(M - \langle M, M\rangle)_t$ is a Q-martingale and $N^{\perp}$ is a martingale under both P and Q, and $(H \cdot \langle M, M\rangle)_t$ is a predictable process.

Proof From Bayes' formula of change of measure in conditional expectation,

$$E_Q(\Delta M_t|\mathcal{F}_{t-1}) = \frac{E_P(Z_t\Delta M_t|\mathcal{F}_{t-1})}{E_P(Z_t|\mathcal{F}_{t-1})} = E_P\big(\Delta M_t\frac{Z_t}{Z_{t-1}}|\mathcal{F}_{t-1}\big) =$$

$$E_P\big(\Delta M_t(1 + \frac{\Delta Z_t}{Z_{t-1}})|\mathcal{F}_{t-1}\big) = E_P(\Delta M_t|\mathcal{F}_{t-1}) + E_P((\Delta M_t)^2|\mathcal{F}_{t-1}) = \ 0 \ + \ \Delta\langle M, M\rangle_t$$

which means that $(M_t - \langle M, M\rangle_t)$ is a Q-martingale.

On the other hand

$$E_Q(\Delta N_t^{\perp}|\mathcal{F}_{t-1}) = E_P(\Delta N_t^{\perp}\Delta M_t|\mathcal{F}_{t-1}) = \Delta\langle N^{\perp}, M\rangle_t = 0$$

since $N^{\perp}$ and M are orthogonal martingales.

In example 3, we compute $\langle M(\theta), M(\theta)\rangle$, and find the law of (ξ_s) under the probability measure $Q_T(\theta)$.
Recall that the characteristic function of the gaussian distribution $\mathcal{N}(\mu,\sigma^2)$ is

$$\varphi_X(u) := E_{\mu,\sigma^2}(\exp(iuX)) = \exp\big(iu\mu - \frac{1}{2}u^2\sigma^2\big)$$

where $X(\omega)$ is $\mathcal{N}(\mu,\sigma^2)$-distributed and i is the imaginary unit.
Now we want to compute the characteristic function of the vector $\xi_1,\dots,\xi_t$ under the measure Q.
We have that for $u=(u_1,\dots,u_t)\in\mathbb{R}^t$

$$E_Q\big(\exp(i\sum_{s=1}^t u_s\xi_s)\big) = E_P\big(Z_t\exp(i\sum_{s=1}^t u_s\xi_s)\big) =$$

$$E_P\left(\exp\left(\sum_{s=1}^t (iu_s+\theta)\xi_s - \frac{1}{2}\theta^2 t\right)\right) =$$

$$E_P\left(\exp\left(\sum_{s=1}^t i(u_s-i\theta)\xi_s + \frac{1}{2}\sum_{s=1}^t (u_s-i\theta)^2\right)\right)\exp\left(\frac{1}{2}\sum_{s=1}^t\{-(u_s-i\theta)^2-\theta^2\}\right) =$$

$$\prod_{s=1}^t E_P\left(\exp\left((u_s-i\theta)\xi_s + \frac{1}{2}(u_s-i\theta)^2\right)\right)\prod_{s=1}^t \exp(i\theta u_s - \frac{1}{2}u_s^2)$$

$$= 1\times E_P(\exp(i(\theta+\xi_s)u_s)$$

this means that the law under Q of ξ_s is the same as the law under P of $(\theta+\xi_s)$, i.e. under Q $(\xi_s : s=1,\dots,t)$ are i.i.d. $\mathcal{N}(\theta,1)$.

6. Martingale predictable representation property

Let M be a P-martingale w.r.t. to a discrete time filtration $\{\mathcal{F}_t : t\in\mathbb{N}\}$.
We say that M has the martingale representation property in the filtration $\mathbb{F}=\{\mathcal{F}_t\}$, if any other bounded $(P,\mathbb{F})$-martingale (X_t) can be represented as a constant plus a martingale transform w.r.t. M

$$X_t = X_0 + (Y\cdot M)_t = X_0 + \sum_{s=1}^t Y_s\Delta M_s$$

where (Y_t) is $\mathbb{F}$-predictable, that is Y_t is $\mathcal{F}_{t-1}$-measurable for all t.
Since X is a bounded martingale, also $\Delta M_s(\omega)$ conditionally bounded given $\mathcal{F}_{s-1}$ is bounded on the set $\{\omega : Y_s(\omega)\neq 0\}$,

$$\Delta M_t(\omega) \le \| \Delta X_t \|_{L^\infty(P)} \big|Y_t(\omega)\big|^{-1}$$

Therefore the M_t is locally bounded, where a localizing sequence is given for example by

$$\tau_n := \inf\big\{t : \| \Delta X_{t+1} \|_{L^\infty(P)} \big|Y_{t+1}(\omega)\big|^{-1} > n\big\}$$

which is a stopping time since $\{\tau_n \le t\} \in \mathcal{F}_t$.
Note that this notation covers also the case of d-dimensional martingales. In such case (Y_s) is a d-dimensional predictable process, and

$$\sum_{s=1}^{t} Y_s \Delta M_s = \sum_{s=1}^{t} \sum_{i=1}^{d} Y_s^{(i)} \Delta M_s(i)$$

Lemma 6.1. *Let (M_t) be a $(P, \mathbb{F})$-martingale.*
(M_t) has the predictable representation property in the $(\mathbb{F})$-filtration if and only if
the only bounded $(P, \mathbb{F})$-martingales (N_t) such that the product $(M_t N_t)$ is a $(P, \mathbb{F})$-martingale are constant.

Proof Assume that the PRP holds for M. Then every bounded martingale N has the form $N_t = (H \cdot M)_t$. If N is such that $(N_t M_t)$ is a martingale, necessarily

$$\Delta(M_t N_t) = M_{t-1}\Delta N_t + N_{t-1}\Delta M_t + \Delta M_t \Delta N_t =$$
$$(M_{t-1}H_t + N_{t-1})\Delta M_t + H_t(\Delta M_t)^2$$

This gives a contradiction, since

$$0 = E(\Delta(M_t N_t)|\mathcal{F}_{t-1}) = H_t E((\Delta M_t)^2|\mathcal{F}_{t-1}) \neq 0$$

with positive probability unless either $\Delta M_t = 0$ or $H_t = 0$. This implies that N_t is constant. The same argument gives the opposite implication.

Theorem 6.1. *In the discrete time setting, M has the martingale representation property in the filtration $\mathbb{F}$ if and only if there are no other martingale measures $Q \sim P$ with bounded density for (M_t), that is if $Q \sim P$, $Z(\omega) = \frac{dP}{dQ}(\omega)$ is essentially bounded and (M_t) is a also a $(Q, \mathbb{F})$-martingale, necessarly $Q = P$.*

Proof For simplicity we set $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Assume that $Q \sim P$. We know that $Z_t = Z_t(Q, P)$ is a $(P, \mathbb{F})$-martingale.
By the predictable representation property,

$$\Delta Z_t = Z_{t-1} H_t \Delta M_t$$

where H_t is $\mathcal{F}_{t-1}$-measurable.
We show that M is not a martingale under Q, unless $H_t = 0$.

$$E_Q(\Delta M_t|\mathcal{F}_{t-1}) = E_P\big(\Delta M_t \frac{Z_t}{Z_{t-1}}\big|\mathcal{F}_{t-1}\big) = E_P\big(\Delta M_t\big(1 + \frac{\Delta Z_t}{Z_{t-1}}\big)\big|\mathcal{F}_{t-1}\big) =$$
$$E_P\big(\Delta M_t\big(1 + H_t \Delta M_t\big)\big|\mathcal{F}_{t-1}\big) = E_P(\Delta M_t|\mathcal{F}_{t-1}) + E_P(H_t(\Delta M_t)^2|\mathcal{F}_{t-1}) =$$
$$0 + H_t E_P((\Delta M_t)^2|\mathcal{F}_{t-1}) \neq 0$$

unless $H_t = 0$ P-a.s. for all t. This means that $Z_t = 1$ for all t and $Q = P$.

Viceversa, suppose that the representation property does not hold for M in the filtration $\mathbb{F}$.
This means that there is some other bounded $(P, \mathbb{F})$-martingale N such that the product $(M_t N_t)$ is a martingale. We can take N satisfying $N_0 = 0$ and

$|N_t| \leq 1$. It is a fact from martingale theory that a bounded martingale (N_t) has almost surely a limit.
Define the measure on $\mathcal{F}_t$

$$dQ_t = \left(1 + \frac{N_t}{2}\right) dP_t = Z_t(\omega) dP_t$$

Note that (Z_t) is a P-martingale with $0 < \frac{1}{2} \leq Z_t(\omega) \leq 3/2$ and $Z_0 = 1$, so that Q_t is a probabilty measure equivalent to P_t on $\mathcal{F}_t$.
We have that

$$M_t Z_t = M_t + \frac{(N_t M_t)}{2}$$

is a P-martingale since (M_t) and $(N_t M_t)$ are P-martingales. This means we have constructed another measure $Q_t \sim P_t$, with $Q_t \neq P_t$ such that (M_t) is a Q-martingale.

Example 4. *Consider a sequence of i.i.d. standard normal random variables (ξ_t) on the probability space $(\Omega, \mathcal{F}, P)$. with the filtration of σ algebrae $\mathcal{F}_t = \sigma(\xi_s : 1 \leq s \leq t)$.*
Define $M_t = \sum_{s=1}^{t} \xi_s$. M_t is a P-martingale, since it has independent increments and centered. M_t is also square integrable, since the increments are gaussian. Note that $\mathcal{F}_t = \sigma(M_s : 1 \leq s \leq t)$.
Note that $\eta_t = (\xi_t^2 - 1)$ are also i.i.d. and centered, and $N_t = \sum_{s=1}^{t} \eta_s$ is also a P-martingale.
It follows that the product $(N_t M_t)$ is a P-martingale,m since $E_P(\xi_t \eta_t) = E_P(\xi_t^3 - \xi_t) = 0$.
The filtration $\{\mathcal{F}_t\}$ generated by (M_t) contains the P-martingale (N_t) which is is orthogonal to (M_t). Neither M or N have the predictable representation property.
We show that there exist an equivalent martingale measure for M. Note that $\Delta N_t = (\xi_t^2 - 1) > -1$ P-almost surely.
Therefore

$$Z_t = \prod_{s=1}^{t} (1 + \Delta N_t) = 1 + \sum_{s=1}^{t} Z_{s-1} \Delta N_s > 0$$

defines an equivalent probability measure $dQ_t = Z_t dP_t$.
By Girsanov theorem, since $(M_t N_t)$ is a P-martingale it follows that also $(M_t Z_t)$ is a P-martingale. But this means that (M_t) is a Q-martingale. So $Q \sim P$ but $Q \neq P$ is another martingale measure for P.
In order to construct a bounded $(P, \{\mathcal{F}_t\})$- martingale we can take the i.i.d. sequence of centered and bounded random variables

$$\varepsilon_t := (\xi_t^2 \wedge 1) - E_P(\xi_t^2 \wedge 1) \in (-1, 1)$$

It follows that

$$E_P(\xi_t\varepsilon_t) = E_P(\xi_t(\xi_t^2 \wedge 1)) - E_P(\xi_t)E_P(\xi_t^2 \wedge 1) =$$
$$E_P(\xi_t \mathbf{1}(|\xi_t| > 1)) + E_P(\xi_t^3 \mathbf{1}(|\xi_t| \le 1)) + 0 = 0$$

since the distribution ξ_t is symmetric around 0.
Therefore for any fixed T, the process stopped at T

$$X_t^T := \sum_{s=1}^{t\wedge T} \varepsilon_s$$

is a bounded P-martingale orthogonal to (M_t).

7. Application to hedging

Consider the finite probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{0,1\}^T$, with $T < \infty$, and $\mathcal{F} = 2^\Omega$, the finite collection of all possible subset, and probability measure satisfies $P(\{\omega\}) > 0$ for all $\omega \in \Omega$.
An history is a vector $\omega = (\omega_1, \dots, \omega_T) \in \Omega$ and denote $\omega^t = (\omega_1, \dots, \omega_t)$ for $t \le T$.
Consider a market with a bank account B_t and a stock price S_t, $t = 0, 1, \dots, T$, adapted to the filtration $\mathbb{F}$ with $\mathcal{F}_t = \sigma(\omega_s, s \le t)$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$
We assume that there are $\{\mathcal{F}_t\}$-**predictable** processes $U_t(\omega) > R_t(\omega) > D_t(\omega) > -1$. $B_0 > 0$ and $S_0 > 0$ are determistic values, and we let

$$B_t = B_0 \prod_{s=1}^t (1 + R_t),$$
$$S_t = S_0 \prod_{s=1}^t (1 + D_t + \omega_t(U_t - D_t))$$

Suppose that $G(\omega)$ is a $\mathcal{F}_t$-measurable contingent claim, and we want to find a self-financing hedging strategy (β_t, γ_t) satisfying

$$V_t = \beta_t B_t + \gamma_t S_t = \beta_{t+1} B_t + \gamma_{t+1} S_t\ .$$

Let $\bar{G}(\omega) = G(\omega)/B_T(\omega)$ the discounted contingent claim.
We show first that there is an unique probability measure Q such that $Q \sim P$ and the discounted process $\bar{S}_t := (S_t/B_t)$ is a Q-martingale.
Once we have shown that Q is the unique martingale measure for $(\bar{S}_t)$ in the filtration $\mathbb{F}$, it follows that every $(Q, \mathbb{F})$ martingale (N_t) has the representation as

$$N_t = N_0 + \sum_{u=1}^t H_u \Delta \bar{S}_u$$

where (H_t) is a $\mathbb{F}$-predictable process. In particular we can take $N_t = E_Q(\bar{G}|\mathcal{F}_t)$, and obtain when $t = T$

$$\bar{G}(\omega) = \frac{G(\omega)}{B_T(\omega)} = E_Q(\bar{G}|\mathcal{F}_T) = E_Q(\bar{G}) + \sum_{t=1}^T \gamma_t \Delta \bar{S}_t$$

where (γ_t) is a $\mathbb{F}$-predictable process.

This gives the unique price $c(G) = E_Q(\bar{G})B_0$ and the hedging strategy for the contingent claim G.

Lets' first compute the martingale measure Q.

$$\Delta \bar{S}_t = \left(\frac{S_t}{B_t} - \frac{S_{t-1}}{B_{t-1}} \right) = \frac{S_{t-1}}{B_{t-1}} \left(\frac{(1 + D_t + (U_t - D_t)\omega_t)}{(1 + R_t)} - 1 \right) =$$
$$\frac{S_{t-1}}{B_{t-1}(1 + R_t)} \big((U_t - D_t)\omega_t - (D_t - R_t) \big)$$

Taking conditional expectation with respect to a measure Q, and imposing the martingale property

$$E_Q(\Delta \bar{S}_t | \mathcal{F}_{t-1}) = \frac{S_{t-1}}{B_{t-1}(1 + R_t)} \big((U_t - D_t) E_Q(\omega_t | \mathcal{F}_{t-1}) - (D_t - R_t) \big) = 0$$

which implies that Q is a martingale measure for $(\bar{S}_t)$ if and only if

$$q_t(\omega^{t-1}) := E_Q(\omega_t | \mathcal{F}_{t-1}) = \frac{(R_t - D_t)}{(U_t - D_t)},$$

where $q_t(\omega^{t-1}) \in (0, 1)$ is a probability since we have assumed that $D_t < R_t < U_t$, P a.s, and it is uniquely determined. We define globally the unique risk-neutral measure Q as follows:

$$Q(\omega) = \prod_{t=1}^{T} q_t(\omega^{t-1})^{\omega_t} (1 - q_t(\omega^{t-1}))^{1-\omega_t}$$

and note that $Q(\{\omega\}) > 0$ for all $\omega \in \Omega$, therefore $Q \sim P$.
We define the basic Q-martingale

$$M_t = \sum_{s=1}^{t} \big(\omega_s - q_s(\omega^{(s-1)}) \big)$$

We write

$$\Delta \bar{S}_t = \frac{S_{t-1}}{B_{t-1}(1 + R_t)} (U_t - D_t)(\omega_t - q_t(\omega^{(t-1)})) = \frac{S_{t-1}}{B_{t-1}(1 + R_t)} (U_t - D_t) \Delta M_t$$

and we can represent ΔM_t in terms of $\Delta \bar{S}_t$:

$$\Delta M_t = \frac{B_{t-1}(1 + R_t)}{S_{t-1}(U_t - D_t)} \Delta \bar{S}_t$$

Next we show how to use the martingale representation to compute the hedging strategy for the contingent claim G.

Definition 6. *If $X(\omega)$ is a $\mathcal{F}_T$-measurable random variable, we define its discrete Malliavin derivative or stochastic gradient at time t w.r.t ω_t as*

$$\nabla_t X(\omega) := X(\omega_1, \dots, \omega_{t-1}, 1, \omega_{t+1}, \dots \omega_T) - X(\omega_1, \dots, \omega_{t-1}, 0, \omega_{t+1}, \dots \omega_T),$$

for $1 \le t \le T$.

Note that in general $\nabla_t X(\omega)$ is not $\mathcal{F}_t$ measurable unless the r.v. $X(\omega) = X(\omega^t)$ is $\mathcal{F}_t$-measurable. In such case $\nabla_t X(\omega)$ is also $\mathcal{F}_{t-1}$-measurable.

In particular the following quantities are $\mathcal{F}_{T-1}$-measurable.

$$\nabla_T G(\omega^{T-1}) = (G(\omega^{T-1},1) + G(\omega^{T-1},0)) \quad ,$$
$$\nabla_T \bar{G}(\omega^{T-1}) = (\bar{G}(\omega^{T-1},1) + \bar{G}(\omega^{T-1},0)) = \frac{1}{B_T(\omega)}(G(\omega^{T-1},1) + G(\omega^{T-1},0))$$
$$= \frac{\nabla_T G(\omega^{T-1})}{B_T(\omega)} \qquad \text{since } B_T(\omega) \text{ is } \mathcal{F}_{T-1}\text{-measurable,}$$

and

$$\nabla_T S_T(\omega^{T-1}) = (S_T(\omega^{T-1},1) + S_T(\omega^{T-1},0)) = S_{T-1}(U_T(\omega^{T-1}) - D_T(\omega^{T-1})) \; .$$
$$\nabla_T \bar{S}_T(\omega^{T-1}) = \frac{1}{B_T}\nabla_T \bar{S}_T(\omega^{T-1})$$

Note also that

$$\Delta \bar{S}_T = (\bar{S}_T - \bar{S}_{T-1}) = \frac{S_{T-1}}{B_T}(U_T - D_T)(\omega_T - q_T) = \nabla_T \bar{S}_T(\omega_T - q_T)$$

so that we can write

$$\Delta M_T = (\omega_T - q_T(\omega^{T-1})) = \frac{1}{\nabla_T \bar{S}_T}\Delta \bar{S}_T = \frac{B_T}{\nabla_T S_T}\Delta \bar{S}_T$$

We have

$$\bar{G}(\omega) = \bar{G}(\omega^{T-1},\omega_T) = \bar{G}(\omega^{T-1},0) + (\bar{G}(\omega^{T-1},1) - \bar{G}(\omega^{T-1},0))\omega_T =$$
$$\bar{G}(\omega^{T-1},0) + \nabla_T \bar{G}(\omega^{T-1})\omega_T =$$
$$\bar{G}(\omega^{T-1},0) + \nabla_T \bar{G}(\omega^{T-1})q_T + \nabla_T \bar{G}(\omega^{T-1})(\omega_T - q_T) =$$
$$E_Q(\bar{G}|\mathcal{F}_{T-1}) + \nabla_T \bar{G}\;\Delta M_T = E_Q(\bar{G}|\mathcal{F}_{T-1}) + \frac{\nabla_T \bar{G}}{\nabla_T S_T}B_T \Delta \bar{S}_T$$
$$= E_Q(\bar{G}|\mathcal{F}_{T-1}) + \frac{\nabla_T G}{\nabla_T S_T}\Delta S_T - \frac{\nabla_T G}{\nabla_T S_T}R_T S_{T-1}$$
$$= E_Q(\bar{G}|\mathcal{F}_{T-1}) + \frac{\nabla_T G}{\nabla_T S_T}\Delta S_T - \frac{\nabla_T G}{\nabla_T S_T}\frac{S_{T-1}}{B_{T-1}}\Delta B_t$$

By investing at time $(T-1)$ the (random) value

$$c_{T-1}(G) = E_Q(\bar{G}|\mathcal{F}_{T-1}(\omega)B_{T-1}(\omega) = \frac{E_Q(G|\mathcal{F}_{T-1})(\omega)}{1+R_T}$$

we replicate the contingent claim G as follows: we buy the amount of stocks

$$\gamma_T = \frac{\nabla_T G}{\nabla_T S_T}$$

at price $\gamma_T S_{T-1}$ (if $\gamma_T < 0$ we short-sell stocks) , if necessary by borrowing from the bank at the predictable interest rate R_T, and buy the amount of

$$\beta_T = \frac{1}{B_{T-1}}\Big(c_{T-1}(G) - \gamma_T S_{T-1}\Big)$$

bonds at price B_{T-1}, so that our capital is

$$V_{T-1} = c_{T-1}(G) = \beta_T B_{T-1} + \gamma_T S_{T-1}$$

At time $(T-1)$ the value of our portfolio is

$$V_{T-1} = \beta_T B_{T-1} + \gamma_T S_{T-1} = c_{T-1}(G)$$

while at time T the value of the portfolio becomes

$$\begin{aligned}
&V_T = \beta_T B_T + \gamma_T S_T = \beta_T B_{T-1}(1+R_T) + \gamma_T S_{T-1} + \gamma_T \Delta S_T \\
&= E_Q(G|\mathcal{F}_{T-1}) - \gamma_T S_{T-1}(1+R_T) + \gamma_T S_{T-1} + \gamma_T \Delta S_T \\
&= E_Q(G|\mathcal{F}_{T-1}) - \gamma_T S_{T-1} R_T + \gamma_T \Delta S_T = \\
&E_Q(G|\mathcal{F}_{T-1}) + \gamma_T (S_T - (1+R_T)S_{T-1}) = E_Q(G|\mathcal{F}_{T-1}) + B_T \gamma_T \Delta \bar{S}_T = G(\omega)
\end{aligned}$$

Remark The martingale measure Q when it is unique gives a device to compute the price and hedging strategy. In fact the price hedging can be computed without using probability, once we have assumed that all histories $\omega \in \Omega$ have positive probability:
A direct way to compute the hedging without using martingales is to solve at time T the system of equations:

$$\begin{aligned}
G(\omega^{T-1}, 0) = B_T \beta_T + \gamma_T S_{T-1}(1+D_T) \\
G(\omega^{T-1}, 1) = B_T \beta_T + \gamma_T S_{T-1}(1+U_T)
\end{aligned}$$

By substracting these two equations we get

$$\gamma_T = \frac{\nabla_T G(\omega^{T-1})}{S_{T-1}(U_T - D_T)}$$

and if the two equations with respective weights $(1 - q_T(\omega^{T-1}))$ corresponding to $\omega_T = 0$ and $q_T(\omega^{T-1})$ corresponding to $\omega_T = 1$ we obtain

$$\begin{aligned}
\beta_T &= \frac{1}{B_T}\big(E_Q(G|\mathcal{F}_{T-1}) - \gamma_T E_Q(S_T|\mathcal{F}_{T-1})\big) \\
&= \frac{1}{B_T} E_Q(G|\mathcal{F}_{T-1}) - \gamma_T \frac{S_{T-1}}{B_{T-1}}
\end{aligned}$$

combining these toghether we get the price of the contingent claim at time $(T-1)$:

$$c_{T-1}(G) = \beta_T B_{T-1} + \gamma_T S_{T-1} = \frac{1}{1+R_T} E_Q(G|\mathcal{F}_{T-1})$$

The martingale method has the advantage that it gives a probabilistic interpretation to the price of the contingent claim, which can be computed directly as a Q-expectation.
The other reason is that the martingale method can be extended to the continuous-time setting.

The price and the hedging strategy in the whole time interval $t = 1, \dots, T$, is then obtained by induction:
Let $c_t(G)$ be the price of the contract G at time $t \le T$. This is a $\mathcal{F}_t$-measurable contingent claim. This means that are able to hedge the contingent claim G expiring at time T if and only if at time t we own a portfolio of value $c_t(G)$. By repeating the martingale argument or by writing directly the system of equations we find the price of the contract at time $(t-1)$ $c_{t-1}(G)$ and the replicating portfolio $\beta_t(\omega^{t-1}), \gamma_t(\omega^{t-1})$.
The advantage the martingale method is that enables to compute directly price and replicating strategy at all times t by computing Q-expectations.
The predictable representation property of the Q-martingale M gives

Theorem 7.1. *Discrete Clarck-Ocone formula:*

$$E_Q(\bar{G}|\mathcal{F}_t)(\omega) = E_Q(\bar{G}) + \sum_{s=1}^{t} \nabla_s E_Q(\bar{G}(\omega)|\mathcal{F}_s)\big(\omega_s - q_s(\omega^{s-1})\big)$$

$$= E_Q(\bar{G}) + \sum_{u=1}^{t} \frac{\nabla_u E_Q(\bar{G}(\omega)|\mathcal{F}_u)}{\nabla_u \bar{S}_u}\, \Delta\bar{S}_u$$

where by definition $\nabla_t E_Q(\bar{G}(\omega)|\mathcal{F}_t)$ is $\mathcal{F}_{t-1}$-measurable.

We set

$$\gamma_t = \frac{\nabla_t E_Q(G(\omega)|\mathcal{F}_t)}{\nabla_t S_t}$$

This gives

$$\begin{aligned} V_t &= E_Q(G|\mathcal{F}_t) = E_Q(G|\mathcal{F}_{t-1}) + \gamma_t B_t \Delta\bar{S}_t \\ &= \frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t} + \gamma_t \Delta S_t + \left(\frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t} - \gamma_t S_{t-1}\right)\frac{1}{B_{t-1}}\Delta B_t \\ &= V_{t-1} + \gamma_t \Delta S_t + \beta_t \Delta B_t \end{aligned}$$

where

$$\beta_t = \left(\frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t} - \gamma_t S_{t-1}\right)\frac{1}{B_{t-1}}$$

This means that to obtain a portfolio with value $E_Q(G|\mathcal{F}_t)$ at time t, we need to invest

$$c_{t-1} := E_Q(G|\mathcal{F}_{t-1})/(1+R_t)$$

at time $(t-1)$. Equivalently, to have $E_Q(G\frac{B_t}{B_T}|\mathcal{F}_t)$ in our portfolio at time t we need to invest the amount

$$E_Q(G\frac{B_{t-1}}{B_T}|\mathcal{F}_{t-1}) \quad \text{at time } (t-1)\ .$$

Inductively , to have $G = E_Q(G|\mathcal{F}_T)$ at time T we have to invest at time $s \le T$ the amount

$$c_t(G) = E_Q(G\frac{B_t}{B_T}|\mathcal{F}_t)$$

at time t.
The hedging at time $(t-1)$ is given by

$$\gamma_t = \frac{\nabla_t E_Q(G(\omega)\frac{B_t}{B_T}|\mathcal{F}_t)}{\nabla_t S_t} = \frac{\nabla_t c_t(G)}{\nabla_t S_t},$$

$$\beta_t = \Big(c_{t-1}(G) - \gamma_t S_{t-1}\Big)\frac{1}{B_{t-1}}$$

giving

$$V_t = c_t(G) = c_0(G) + \sum_{u=1}^{t}\big(\gamma_u \Delta B_u + \beta_u \Delta B_u\big)$$

$$V_T = G = c_0(G) + \sum_{u=1}^{T}\big(\gamma_u \Delta B_u + \beta_u \Delta B_u\big)$$

When R_t is deterministic, we can take the discounting factors B_t/B_T outside the conditional expectation.

If (D_t, R_t, U_t) are all deterministic, then under the martingale measure Q the random variables ω_t is independent from the past. Then the computation of the hedging strategy may be simplified by using the following formula:

Corollary 7.1. *If (D_t, R_t, U_t) are deterministic at all $t \le T$, conditional expectation and gradient commute in Ito-Clarck formula*

$$\nabla_t E_Q(G|\mathcal{F}_t) = E_Q(\nabla_t G|\mathcal{F}_t) = E_Q(\nabla_t G|\mathcal{F}_{t-1}) \ ,$$

giving

$$E_Q(G|\mathcal{F}_t)(\omega) = E_Q(G) + \sum_{s=1}^{t} E_Q(\nabla_s G|\mathcal{F}_s)\big(\omega_s - q_s(\omega^{s-1})\big) \ .$$

Proof When $\omega = (\omega_1, \dots, \omega_T)$ we denote $\omega^{t,T}$ the vector $(\omega_t, \dots, \omega_T)$. Using the independence of the r.v. (ω_t),

$$E_Q(\nabla_t G|\mathcal{F}_t)(\omega_t) = \sum_{\omega^{t+1,T} \in \{0,1\}^{T-t}} \big\{G(\omega^{t-1}, 1, \omega^{t+1,T}) - G(\omega^{t-1}, 0, \omega^{t+1,T})\big\} Q(\omega^{t+1,T})$$
$$= \nabla_t E_Q(G|\mathcal{F}_t)(\omega_t)$$

which is $\mathcal{F}_{t-1}$-measurable.

Example 5. *Assume that $R_t = r, U_t = u, D_t = d$ deterministic, with $-1 < d < r < u$. Then $q_t = q = (r-d)/(u-d)$ is constant. We have that*

$$S_t = S_0(1+u)^{N_t}(1+d)^{t-N_t}$$

where $N_t = \sum_{s=1}^{t} \omega_s$.
Then if $G(\omega) = \varphi(S_T)$ is a plain european option, we compute the price at time $t = 0$ using the distribution Binomial(q, T).

$$V_0 = c_0(G) = B_0 E_Q(\varphi(S_T)/B_T) =$$
$$(1+r)^{-T} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} \varphi\big(S_0(1+u)^n(1+d)^{T-n}\big) \ .$$

Similarly since the conditional distribution of $(N_T - N_t)$ given $\mathcal{F}_t$ is Binomial$(q, T-t)$, at time t the price of the replicating portfolio is

$$V_t = c_t(G) = B_t E_Q(\varphi(S_T)/B_T|\mathcal{F}_t) =$$
$$(1+r)^{t-T} \sum_{n=0}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n} \varphi\big(S_0(1+u)^{N_t+n}(1+d)^{T-N_t-n}\big) \ .$$

with this amount of money, we invest in γ_{t+1} *stocks and invest the rest in the bank account, with*

$$\gamma_{t+1} = \frac{\nabla_{t+1} c_{t+1}(G)}{\nabla_{t+1} S_{t+1}} = (1+r)^{t+1-T} \frac{E_Q(\nabla_{t+1} G | \mathcal{F}_t)}{S_t(u-d)} =$$

$$(1+r)^{t+1-T} \frac{1}{S_t(u-d)} \sum_{n=0}^{T-t-2} \left\{ \binom{T-t-2}{n} q^n (1-q)^{T-t-2-n} \times \right.$$

$$\left. \times \Big(\varphi\big(S_0(1+u)^{N_t+n+1}(1+d)^{T-N_t-n-2}\big) - \varphi\big(S_0(1+u)^{N_t+n}(1+d)^{T-N_t-n-1}\big) \Big) \right\}$$