Inverse problems course, spring 2014   Exercise 3 (February 4-7, 2014)
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Note that this exercise has more than one page and contains several types of exercises.

Please complete the theoretical exercises (marked with T) before the exercise session and be prepared to present your solution there.

T1. Denote by $\mathcal{R}f(s, \vec{\theta})$ the Radon transform of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Consider the forward map $F : f \mapsto \mathcal{R}f$ defined on a model space of well-behaving functions, for example essentially bounded functions supported in the unit disc. Prove that F is injective. (Hint: use the Fourier slice theorem.)

Answer

Let

$$B_D := \{f : \mathbb{R}^2 \to \mathbb{R} : f \text{ is continuous and } \mathrm{supp} f \subset \overline{B(0,1)} \subset \mathbb{R}^2\}.$$

i.e. B_D is the set bounded functions which are supported in unit disc. We want to prove following:

Claim: Radon transform $\mathcal{R} : B_D \to B_D$ is well defined and injective.
Let $f \in B_D, \theta \in S^1$ and $s > 0$. Consider line $L_s := \{x \in \mathbb{R}^2 : x \cdot \theta = s\}$ of integreation in $\mathcal{R}f(\theta, s)$. Define $D = \overline{B(0,1)}$.

We first show that $L_s \cap D = \emptyset$ if $s > 1$ which proves that $\mathcal{R}f(\theta, s) = 0$ since f is supported in D. Suppose that $\theta_2 \neq 0$ and $x \in L_s$. Then it holds that $x \neq 0$ and

$$x \cdot \theta = s \Leftrightarrow x_1\theta_1 + x_2\theta_2 = s \Leftrightarrow x_2 = \frac{s - x_1\theta_1}{\theta_2}.$$

Calculate

$$\frac{d}{dt}||x||^2 = 2\frac{x_1 - \theta_1 s}{\theta_2^2}$$

$$\text{if } \frac{d}{dt}||x||^2 = 0 \Rightarrow x_1 = s\theta_1 \text{ and } x_2 = \frac{s - s\theta_1^2}{\theta_2} = s\theta_2.$$

Therefore it holds that $\mathrm{dist}(L_s, 0) \geq s$ and if $s > 1 \Rightarrow L_s \cap D = \emptyset$.
Let

$$\gamma_{\theta,s}(t) = (t, \frac{s - x_1\theta_1}{\theta_2}).$$

Now it holds that

$$\mathcal{R}f(\theta, s) = \int_{x \cdot \theta = s} f(x) dx^\perp = \int_{-\infty}^{\infty} f(\gamma_{\theta,s}(t)) dt.$$

Since curve $\gamma_{\theta,s}$ is continuous and f is also continuous we know that the composite mapping $f \circ \gamma$ is continuous. Next calculation shows that the mapping $f \circ \gamma_{\theta,s}$ is integrable and that Radon transform is well defined

$$|\mathcal{R}f(\theta, s)| = |\int_{-\infty}^{\infty} f(\gamma_{\theta,s}(t))dt| \leq ||f||_{\infty} m_1(L_S \cap D) \leq 2||f||_{\infty}.$$

Here m_1 is the one dimensional Lebesque measure. Notice that we have now proven that $\mathcal{R}f \in B_D$.

Next we consider the injectivity of Radon transform. Let f, g. Notice that Radon tranform is a linear mapping from vectorspace B_D into B_D. We use the Fourier slice theorem which is Theorem 2.3.1. in textbook and basic properties of Fourier transform. Suppose that

$$0 = \mathcal{R}f - \mathcal{R}g = \mathcal{R}(f - g).$$

By slice theorem it then holds that

$$0 = \int_{-\infty}^{\infty} \underbrace{\mathcal{R}(f-g)(\theta, s)}_{=0 \text{ almost everywhere}} e^{-isr}ds = \widehat{f-g}(r\theta).$$

Since Fourier transform is one-to-one, it holds that $f(x) - g(x) = 0$ almost everywhere. Therefore we have proven, that $f = g \in B_D$.

T2. Calculate through the proof of Lemma L 1 on page 11 of Professor Erkki Somersalo's classical material uploaded to the course web page by the name *Somersalo_Inversio_Liite2.pdf*. I know it's in Finnish, but it's mostly mathematical formulas so it should not matter. Alternatively, you can look for another source in the literature. The main point is to understand why the Fourier transform of a Gaussian is also a Gaussian.

Answer

Claim: Let
$$\psi : \mathbb{R}^n \to \mathbb{R}, \ \psi(x) = (2\pi)^{n/2} e^{-||x||^2/2}.$$
For function ψ it is true, that
$$\widehat{\psi}(\xi) = e^{-||\xi||^2/2} \text{ and } \mathcal{F}^{-1}\widehat{\psi}(x) = \psi(x).$$

Proof: We first consider one dimensional case. Define function
$$H : \mathbb{R} \to \mathbb{R}, \ H(s) := \int_{\mathbb{R}} e^{-its} e^{-t^2/2}dt.$$

Since H is nice enough we can calculate

$$\frac{d}{ds}H(s) = \int_{\mathbb{R}} \frac{d}{ds}(e^{-its})e^{-t^2/2}dt = -i\int_{\mathbb{R}} te^{-its}e^{-t^2/2}dt = i\int_{\mathbb{R}} e^{-its}\frac{d}{dt}e^{-t^2/2}dt.$$

Next we use the integration by parts formula

$$\frac{d}{ds}H(s) = i\int_{\mathbb{R}} e^{-its}\frac{d}{dt}e^{-t^2/2}dt = \underbrace{i[e^{-its}e^{-t^2/2}]_{-\infty}^{\infty}}_{=0} - i\int_{\mathbb{R}}\frac{d}{dt}(e^{-its})e^{-t^2/2}dt$$

$$= -s\int_{\mathbb{R}} e^{-its}e^{-t^2/2}dt = -sH(s).$$

Therefore we have obtained a separable differential equation $H'(s) = -sH(s)$. It is easy to prove that

$$H(s) = H(0)e^{-s^2/2}.$$

In order to find $H(0)$ we have to find it from the original deffinition of H.

$$H(0)^2 = \left(\int_{\mathbb{R}} e^{-t^2/2}dt\right)^2 = \int_{\mathbb{R}} e^{-x^2/2}dx \int_{\mathbb{R}} e^{-y^2/2}dy = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)/2}dxdy.$$

We can find $H(0)$ easier, if move to Spherical coordinates. Recall that the spherical coordinates in $\mathbb{R}^2$ have a following definition

$$\begin{cases} x = r\cos(\phi) \\ y = r\sin(\phi) \end{cases} \Rightarrow dxdy = rdrd\phi.$$

Then we just complete calculation

$$H(0)^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}rdrd\phi = -2\pi\int_0^{\infty}\frac{d}{dr}e^{-r^2/2}dr = 2\pi$$

$$\Rightarrow H(s) = \sqrt{2\pi}\, e^{-s^2/2}.$$

Now we move to $n-$dimensional case. Using the definition of Fourier transform and 1-dimensional case it is now easy to calculate

$$\widehat{\psi}(\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (2\pi)^{n/2}e^{-||x||^2/2}e^{-ix\cdot\xi}dx = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-(x\cdot x/2+ix\cdot\xi)}dx$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-\sum_{k=1}^n(x_k^2/2+ix_k\xi_k)}dx = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\prod_{k=1}^n e^{-(x_k^2/2+ix_k\xi_k)}dx$$

$$= \frac{1}{(2\pi)^{n/2}}\prod_{k=1}^n\int_{\mathbb{R}} e^{-(x_k^2/2+ix_k\xi_k)}dx_k = \frac{1}{(2\pi)^{n/2}}\prod_{k=1}^n H(\xi_k)$$

$$= \frac{1}{(2\pi)^{n/2}}\prod_{k=1}^n \sqrt{2\pi}\, e^{-\xi_k^2/2} = e^{-||\xi||^2/2}.$$

If we recall the formula for inverse Fourier transform and note that ψ is symmetric we also get the second claim from last monster equation.

T3. Consider the inverse problem defined by the measurement model $m = Af$ in the cases

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $m = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, (b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 13 & 31 \end{bmatrix}$, $m = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Which of Hadamard's conditions are violated, if any?

Answer: In case (a) we note that matrix A has a nontrivial kernel. Therefore we cannot find any continous inverse for A since A is a square matrix. This violates every Hadamard's conditions.

Let $x \in \mathbb{R}^2 \backslash \{0\}$. Then it holds that

$$Ax = \begin{bmatrix} x_2 \\ x_1 \\ 13x_1 + 31x_2 \end{bmatrix} \neq 0.$$

This shows that A is injective. But still equation

$$Ax = \begin{bmatrix} x_2 \\ x_1 \\ 13x_1 + 31x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

does not have a solution, since by equation (1) it must be that $x_1 = x_2 = 1$, and then it should be that $m_3 = 44$.

If $m = \begin{bmatrix} 1 \\ 1 \\ 44 \end{bmatrix}$, there would still be a problem with Hadamards conditions 2 and 3 since $\dim \mathrm{Im} A = 2 < 3 = \dim \mathbb{R}^3$. Therefore it holds that for inctance matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a smooth inverse of A. Since there exists a invere of A it holds that H1 is true. On the other hand B is not only matrix that is an inverse of A and for matrix B even the smallest maesurement error ruins the recovery. So H2 and H3 do not hold.

T4. Recall problem T2 from Exercise 1.

(a) Construct the 10×10 convolution matrix A with the property that Af is the same vector than $\widetilde{p} * f$.

(b) Check that $A^T = A$.

Answer: Let $(\widetilde{p} * f) \in \mathbb{R}^{10}$ be the discrete convolution vector. Keeping the periodic boundary condition in mind we recall that j^{th} entry of $(\widetilde{p} * f)$ is

$$(\widetilde{p} * f)_j = \sum_{k=-2}^{2} \widetilde{p}_k f_{j-k}. \tag{2}$$

Since we had that $\widetilde{p} = [1, 1, 1, 1, 1]^T$ we see from equation (2) that matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

It is now easy to see that $A^T = A$ since A is symmetric near main diagonal and there is 3 number 1's in simmillar positions in both right uppercorner and left bottomcorner.

You can work on these Matlab exercises (marked with M) in the exercise session.

M1. Consider the linear equation $x_1 + x_2 = 1$.

    (a) Write the equation in the matrix form $Ax = y$. (That is, specify the elements in the 1×2 matrix A and the 1-vector $y \in \mathbb{R}$.)

    (b) Use Matlab to compute the singular value decomposition $A = UDV^T$.

    (c) Using the result of (b), construct D^+ and the minimum norm solution $x^+ := VD^+U^Ty$ in Matlab. Draw the line $x_2 = -x_1 + 1$ and the point x^+ in the (x_1, x_2)-plane. Discuss the result.

Answer:

Here is an example script (there are, of course, many ways to do the same things):

```
% Form matrix A and vector y
A = [1,1];
y = 1;

% Compute the SVD of A
[U,D,V] = svd(A);

% Construct D_plus using the SVD and a for loop (there are plenty of ways
% doing the same thing, and applying function pinv.m directly to matrix A
% would even make the separate computation of the SVD above useless)
D_plus = zeros(size(D'));
for ii=1:min(size(D))
    if D(ii,ii) > 0
        D_plus(ii,ii) = 1/D(ii,ii);
    end
end

% Compute the minimum norm solution
x_plus = V*D_plus*U.'*y;

% Create a grid on the interval [0,1] for plotting the line x2=-x1+1
x1 = linspace(0,1);

% Plot the line x2=-x1+1 and the point x_plus
figure
plot(x1,-x1+1)
hold on
plot(x_plus(1),x_plus(2),'o')
```

As expected, the minimum norm solution x^+ is that point on the line $x_2 = -x_1 + 1$ that has the smallest norm (smallest distance from the origin).

M2. Consider equations $x_1 + x_2 = 1$, $x_2 = -2$ and $-\frac{1}{3}x_1 + x_2 = -2$.

(a) Write the equations in the matrix form $Ax = y$. (That is, specify the elements in the 3×2 matrix A and the vector $y \in \mathbb{R}^3$.)

(b) Use Matlab to compute the singular value decomposition $A = UDV^T$.

(c) Using the result of (b), construct D^+ and the minimum norm solution $x^+ := VD^+U^Ty$ in Matlab. Draw the three lines specified by the equations and the point x^+ in the (x_1, x_2)-plane. Discuss the result.

Answer:

Here is an example script (once again, there are many ways to do the same things):

```
% Form matrix A and vector y
A = [1,1;0,1;-1/3,1];
y = [1;-2;-2];

% Compute the SVD of A
[U,D,V] = svd(A);

% Construct the D_plus using the SVD as in M1
D_plus = zeros(size(D'));
for ii=1:min(size(D))
    if D(ii,ii) > 0
        D_plus(ii,ii) = 1/D(ii,ii);
    end
end

% Compute the minimum norm solution
x_plus = V*D_plus*U.'*y;

% Create a grid on the interval [-3,3] for plotting the lines x2=-x1+1,
% x_2=-2 and x2=x1/3-2
x1 = linspace(-3,3);
figure
plot(x1,-x1+1,'LineWidth',1)
hold on
plot(x1,-2,'LineWidth',1)
plot(x1,x1/3-2,'LineWidth',1)
plot(x_plus(1),x_plus(2),'o')
```

In this case it is more difficult to interpret the point x^+ in the figure. However, theory says that it is the point that

1. is a least squares solution of $Ax = y$, (i.e. it has, in the least squares sense, the smallest distance to the three lines) and

2. has the smallest norm (distance to the origin).

M3. Load the function `Radon_example.m` from the course website. Add various amounts of noise to the sinogram and examine the effet of different filters. You can see the possible filter choices by typing `help iradon` in Matlab.

Answer:

Let us first modify the file a bit to make it more suitable for doing this exercise:

```
im = phantom(256);
ang = [1:180];

% Set noise level
noiselevel = 0.1;

R = radon(im,ang);
R = R + noiselevel*max(abs(R(:)))*randn(size(R));
backp = iradon(R,ang,'none');
recon = iradon(R,ang,'Ram-Lak');
recSL = iradon(R,ang,'Shepp-Logan');
recCS = iradon(R,ang,'Cosine');
recHM = iradon(R,ang,'Hamming');
recHN = iradon(R,ang,'Hann');

% Parameter for adjusting the contrast in the images (if the images have
% negative elements, then gamma < 1 will lead to complex values in the
% images and imshow.m will give warnings but you don't have to care about
% these)
gamma = 0.5;

% Plot all the images in the same figure
figure(1)
clf
subplot(2,4,1)
imshow(im.^gamma)
axis equal
colormap gray
title('Original image')
subplot(2,4,5)
imagesc(R)
axis equal
axis off
colormap gray
```

```
title('Noisy sinogram')
subplot(2,4,2)
imagesc(backp.^gamma)
axis equal
axis off
title('Back-projection without filtering')
colormap gray
subplot(2,4,3)
imshow(recon.^gamma)
axis equal
colormap gray
title('Ram-Lak')
subplot(2,4,4)
imshow(recSL.^gamma)
axis equal
colormap gray
title('Shepp-Logan')
subplot(2,4,6)
imshow(recCS.^gamma)
axis equal
colormap gray
title('Cosine')
subplot(2,4,7)
imshow(recHM.^gamma)
axis equal
colormap gray
title('Hamming')
subplot(2,4,8)
imshow(recHN.^gamma)
axis equal
colormap gray
title('Hann')
```

Testing this file e.g. with noise levels 0, 0.01 and 0.1 seems to indicate that the noise is most efficiently filtered by Hamming and Hann filters. On the other hand, least filtering is given by Ram-Lak.

M4. **Numerical evaluation of the Fourier coefficients of our PSF.** Recall the (here non-normalized) point spread function

$$p(x) = (x + a)^2 (x - a)^2, \text{ for } -a \leq x \leq a,$$

which is implemented in the Matlab file `DC_PSF.m`. The goal of this exercise is to observe the decay of the Fourier coefficients

$$\widehat{p}(n) := \frac{1}{2\pi} \int_{-a}^{a} e^{-inx} p(x) dx, \qquad n > 0,$$

when $n \to \infty$. Use a numerical quadrature (for example Simpson's rule) to compute $\widehat{p}(n)$ approximately for $n = 1, 2, 3, \ldots, N$. Try to go up to, for

example, $N = 32$. Note that when the frequency n grows, you will need more quadrature points in the integration to achieve acceptable accuracy. The point her is to examine how fast the coefficients approach zero as n grows; thus the normalization of the point spread function plays no role (why is this?). What can you deduce about the ill-posedness of the inverse problem *Given* $m = p * f + \epsilon$, *find f?*

Answer:

Here is an example script (using composite Simpson's rule for numerical integration):

```
% Parameter a of the PSF (note that it should satisfy 0<a<1/2)
a = 0.001;

% Create a uniform grid of NN points for x (note that NN should be an odd
% number in order to use (composite) Simpson's rule)
NN = 20001;
x  = linspace(-a,a,NN);

% Choose N (number of Fourier coefficients to be computed)
N = 128;

% Form the weights of the Simpson's rule
weights = zeros(1,NN);
weights(1) = 1;
weights(NN) = 1;
weights(2:2:NN-1) = 4;
weights(3:2:NN-2) = 2;

% Compute the Fourier coefficients using (composite) Simpson's rule
h       = 2*a/(NN-1);
pn = zeros(N,1);
for n=1:N
    pn(n) = h/(2*pi)*sum(weights.*exp(-1i*n*x).*DC_PSF(x,a));
end

% Plot the (absolute values of the) Fourier coefficients
figure
plot(1:N,abs(pn))
xlabel('$$n$$','Interpreter','Latex','FontSize',20)
ylabel('$$\hat{p}(n)$$','Interpreter','Latex','FontSize',20)
```

Running this script e.g. with $a = 0.1$, $a = 0.01$ and $a = 0.001$ illustrates that the Fourier coefficients $\hat{p}(n)$ tend to zero faster for a larger parameter a (as $n \to \infty$). The normalization (or lack of it) plays no role here since it would

just mean a multiplication of each $\hat{p}(n)$ by a constant not depending on n and hence it would have no effect on the speed at which the coefficients $\hat{p}(n)$ tend to zero.

In terms of the inverse problem *"given $m = p * f + \epsilon$, find f"* this means that the problem is ill-posed especially for large parameters a. Namely, note that because $\widehat{p * f} = \hat{p}\hat{f}$, we have

$$\widehat{f}(\xi) = \frac{\widehat{m_\epsilon}(\xi)}{\widehat{p}(\xi)}, \tag{3}$$

where $m_\epsilon = m - \epsilon$. Now, if $\widehat{p}(\xi)$ is small for large ξ's then even small-amplitude high-frequency components (noise) in m_ϵ get amplified, i.e. multiplied by a large number. Then, after taking the Fourier inverse transform, we get a probably useless reconstruction f with high oscillations.