Inverse problems course, spring 2014 Exercise 2 (January 28-31, 2014)
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Note that this exercise has more than one page and contains several types of exercises.

Please complete the theoretical exercises (marked with T) before the exercise session and be prepared to present your solution there.

T1. Assume that the $n \times n$ matrix $U : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal: $UU^T = I = U^T U$. Show that $\|U^T y\| = \|y\|$ for any $y \in \mathbb{R}^n$.

T2. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a linear mapping. Let $(e_i)_{i=1}^k$ be a base of $\mathbb{R}^k$ and $(f_i)_{i=1}^n$ be a base of $\mathbb{R}^n$. Let $[A_{ij}]$ be the matrix of linear mapping A with respect to bases $(e_i)_{i=1}^k$ and $(f_i)_{i=1}^n$ i.e.

$$[A_{ij}](v_1, \ldots, v_k) = A(v), \quad \text{for all } v \in \mathbb{R}^k.$$

Show that there exists such an ortonormal bases $(\widetilde{e}_i)_{i=1}^k$ and $(\widetilde{f}_i)_{i=1}^n$ that matrix $[A_{ij}]$ has form

$$[A_{ij}] = \begin{pmatrix} \sigma_1 & 0 & \ldots & & \\ 0 & \ddots & 0 & & \\ \vdots & 0 & \sigma_l & 0 & \\ & & 0 & 0 & \\ & & & & \ddots \end{pmatrix} \tag{1}$$

where $l = \min\{k, n\}$ and $\sigma_i > 0$ for every $i \in \{1, \ldots, l\}$.

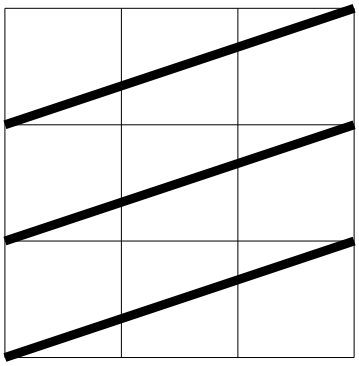
Equivalently. For any real $k \times n$-matrix $A \in M(\mathbb{R}, k, n)$ there exists ortongonal matrices $U \in M(\mathbb{R}, k, k)$ and $V \in M(\mathbb{R}, n, n)$ s.t.

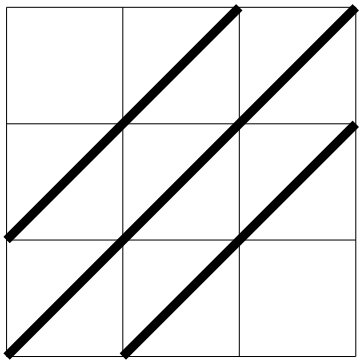
$$A = UDV^T.$$

Here $D \in M(\mathbb{R}, k, n)$ and has the same form as $[A_{ij}]$ in equation (1). Recall that an invertible square matrix $O \in M(\mathbb{R}, n, n)$ is ortogonal iff $O^T = O^{-1}$.

Tip
Recall the Spectral theorem of self-adjoint (symmetric) linear mappings in a finite dimmensional vector spaces and notice that $A^T A : \mathbb{R}^k \to \mathbb{R}^k$ is a self-adjoint linear mapping. You can for instance use book "Linear Algebra" by Peter Petersen as a reference. You should find a copy of it easy by googling!

T3. Thin lines depict pixels and thick lines X-rays in this image:





Give a numbering to the nine pixels ($f \in \mathbb{R}^9$) and to the six X-rays ($m \in \mathbb{R}^6$), and construct the matrix A for the measurement model $m = Af$. The length of the side of a pixel is one.

You can work on these Matlab exercises (marked with M) in the exercise session.

M1. Run the file `DC1_cont_data_comp.m`. Set the number of discretization points to 100 by writing `n = 100;` on line 12 in the file `DC2_discretedata_comp.m`. Then run the file `DC2_discretedata_comp.m` to produce discrete convolution data.

Compute the singular value decomposition of the 100×100 measurement matrix A using the code in the file `DC4_truncSVD_comp.m`. Determine the *condition number* of A, defined as the first (largest) singular value divided by the last (smallest) singular value.

Repeat the above with resolutions $n = 200, 300, 400$. What happens to the condition number when n grows?

M2. Download from the course website all the Matlab routines whose name starts with `XRM`.

(a) Run in Matlab the routines `XRM1_matrix_comp.m`, `XRM2_naive_comp.m`, `XRM3_NoCrimeData_comp.m`, `XRM4_naive_comp.m`, `XRM5_SVD_comp.m` and `XRM6_truncSVD_comp.m`. Choose $N = 32$ in all of these files, leading to reconstruction size 32×32.

(b) Compute the condition number $\mathrm{cond}(A) := d_1/d_{\min\{k,n\}}$ of the measurement matrix A. You get the singular values from `XRM5_SVD_comp.m`.

(c) Record the minimum relative reconstruction error you see when running `XRM6_truncSVD_comp.m`.

M3. Computational study of limited angle tomography with 90° view angle.

(a) In routine `XRM1_matrix_comp.m`, the projection directions are evenly distributed in the interval $[0, 180^\circ)$, excluding the value 180°. Modify line 23 so that the projection directions are evenly distributed in the interval $[0, 90^\circ)$. Run `XRM1_matrix_comp.m`.

(b) Run `XRM2_naive_comp.m`, `XRM3_NoCrimeData_comp.m`, `XRM4_naive_comp.m` and `XRM5_SVD_comp.m`. Compute $\mathrm{cond}(A)$ and compare the result to that of M2(b).

(c) Run `XRM6_truncSVD_comp.m` and record the minimum relative reconstruction error. Compare the result to that of M2(c). How do the reconstructions and singular vectors differ from those you saw in M2(c)?