Inverse Problems
(and their computational solution)

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The Legendary Course
University of Helsinki, spring 2014

This my industrial-academic background

- 1999: PhD, Helsinki University of Technology, Finland
- 2000: R&D scientist at Instrumentarium Imaging
- 2002: Postdoc at Gunma University, Japan
- 2004: R&D scientist at GE Healthcare
- 2005: R&D scientist at Palodex Group
- 2006: Professor, Tampere University of Technology, Finland
- 2009: Professor, University of Helsinki, Finland

Finnish Centre of Excellence in Inverse Problems Research

OULUN YLIOPISTO UNIVERSITY of OULU

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TAMPERE UNIVERSITY OF TECHNOLOGY

UNIVERSITY OF HELSINKI

http://wiki.helsinki.fi/display/inverse/Home

Practical course information

Period III and partly IV:

Lectures
Tuesday 10-12 (Exactum D123),
Wednesday 12-14 (Exactum D123),
Friday 12-14 (Exactum B120).
Lecturer: Professor Samuli Siltanen

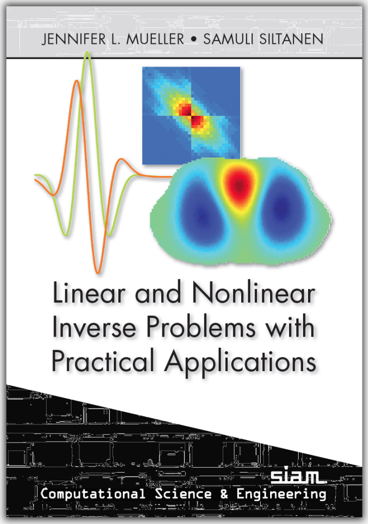
Exercises
Time is not yet decided
Teaching assistants: Esa Niemi & Teemu Saksala

Final exam
Passing the final exam and completing enough exercises corresponds to 10 credit units.

Period IV:

Project work
Computational inversion projet done in teams of two students. Results are reported in the form of a poster on a specific day (announced later).

The project work corresponds to 5 credit units.



All Matlab codes freely available on a website!

Outline

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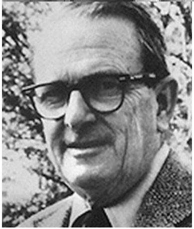
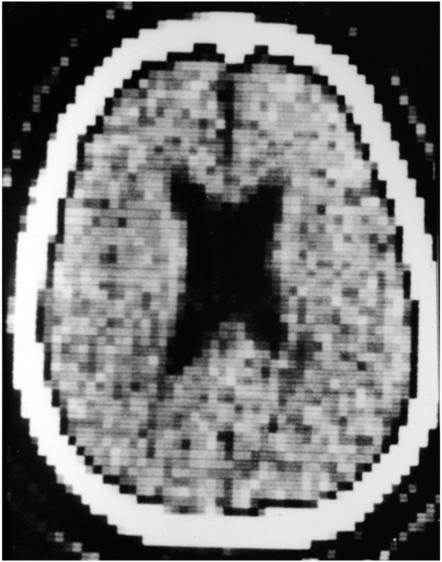
Outline

Godfrey Hounsfield and Allan McLeod Cormack were the first to develop X-ray tomography



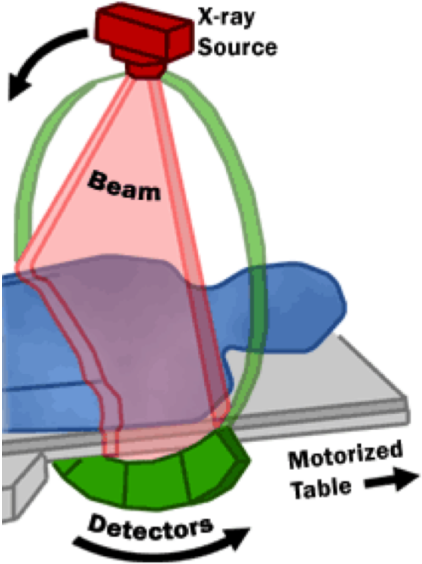
Cormack (left) and Hounsfield (top) received Nobel prizes in 1979. Right: an early tomographic image.



Reconstruction of a function from its line integrals was first invented by Johann Radon in 1917

$$f(P) = -\frac{1}{\pi}\int_0^\infty \frac{d\overline{F_p}(q)}{q}$$

Traditional X-ray tomography requires many projection images using small angular steps



$$\frac{1}{4\pi^2}\int_{S^1}\int_{\mathbb{R}} \frac{\frac{d}{ds}(Rf)(\theta, s)}{x\cdot\theta - s}\, ds\, d\theta$$

Outline

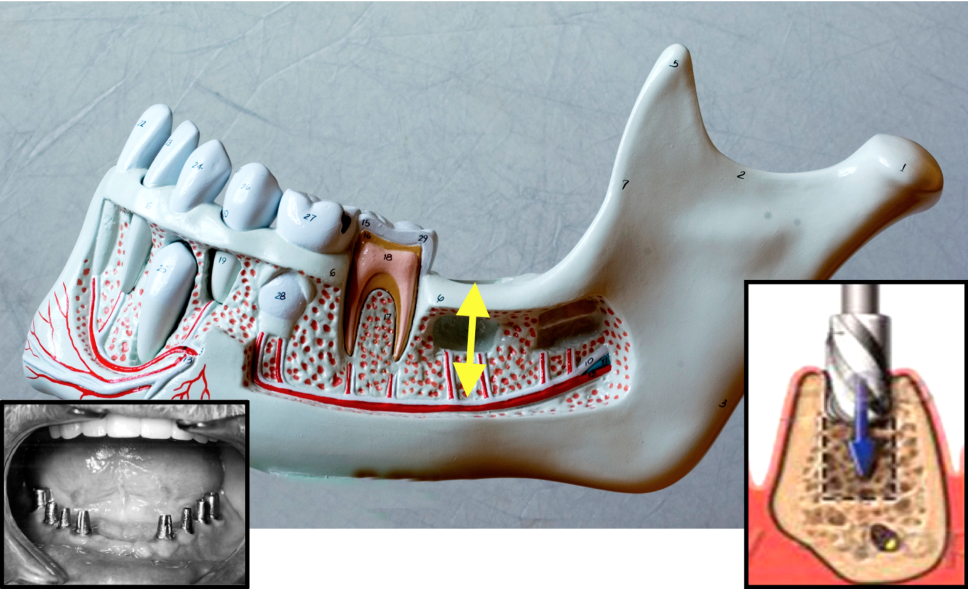
A series of projects started in Finland in 2001, aiming for a new type of low-dose 3D imaging

The goal was a mathematical algorithm with

Input: small number of digital X-ray images from any X-ray device.
Output: Three-dimensional reconstruction with high enough quality for the clinical task at hand.

Products of Instrumentarium Imaging in 2001:

Application: dental implant planning, where a missing tooth is replaced with an implant



Nowadays, a digital panoramic imaging device is standard equipment at dental clinics

A panoramic dental image offers a general overview showing all teeth and other dento-maxillofacial structures simultaneously.

Panoramic images are not suitable for dental implant planning because of unavoidable geometric distortion.

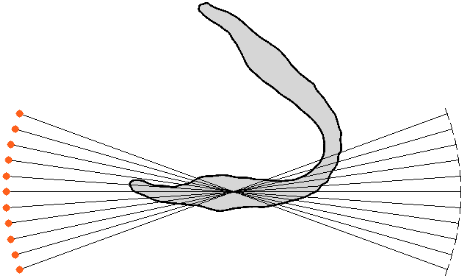
We reprogram the panoramic X-ray device so that it collects projection data by scanning

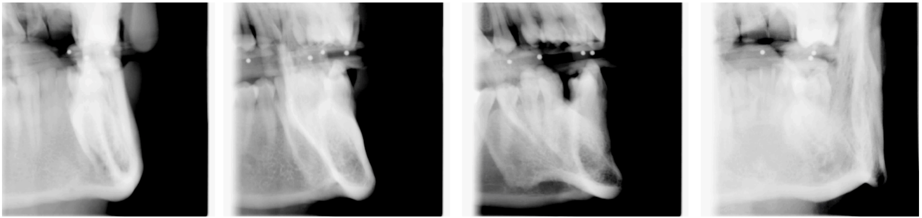
Number of projection images: **11**

Angle of view: **40 degrees**

Image size: **1000×1000 pixels**

The detector of a panoramic device is very narrow, so images are formed using a scanning movement analogously to a xerox machine.





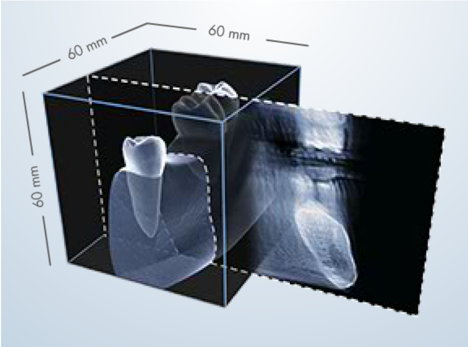
This low-dose 3D imaging technique has been commercialized by Palodex Group

The VT device has been available in the market from year 2007.

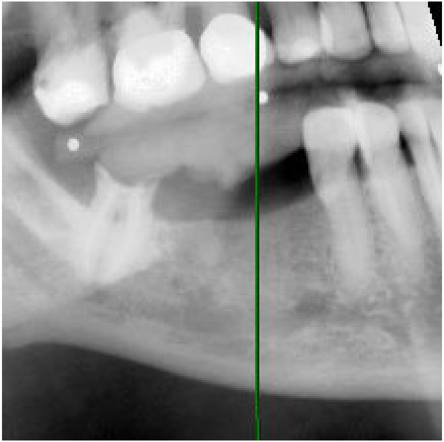
Remarkably, a digital panoramic imaging device, that the dental clinic already has, becomes a 3D imaging machine by a software update.

The core of that software update is a computational inversion algorithm.

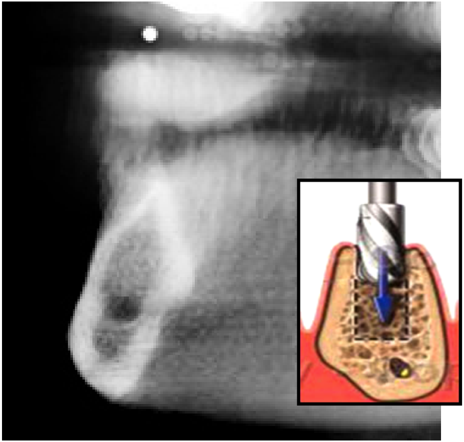
See **http://www.vt-cube.com/**



Here are example images of an actual patient. Left: navigation image. Right: desired slice.



Kolehmainen, Vanne, S, Järvenpää, Kaipio, Lassas & Kalke **2006**, Kolehmainen, Lassas & S **2008**



Cederlund, Kalke & Welander **2009**, Hyvönen, Kalke, Lassas, Setälä & S **2010**, U.S. patent 7269241

The radiation dose of the VT device is the lowest among 3D dental imaging modalities

Modality	μSv
Head CT	2100
CB Mercuray	558
i-Cat	193
NewTom 3G	59
VT device	**13**

Ludlow, Davies-Ludlow, Brooks & Howerton **2006**

Outline

Inverse problem: ozone layer tomography

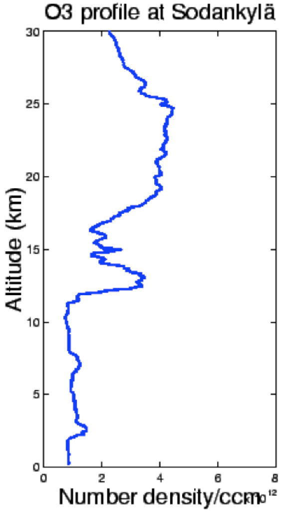
Direct problem:

If the ozone profile of the atmosphere were known, what star occultation measurements would we get?

Inverse problem:

Given star occultation measurements , what is the ozone profile?

As a result we get ozone density as function of altitude



This inverse problem is mathematically the same than the CT problem, except with limited data

Sources:
European Space Agency
Finnish Meteorological Institute
Envisat and GOMOS projects
http://www.fmi.fi/tutkimus_otsoni/otsoni_26.html
http://envisat.esa.int/handbooks/gomos/CNTR2.htm

Outline

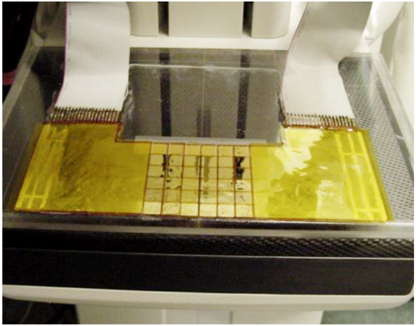
We recently started to study EIT for imaging changes in vocal folds due to excessive voice use

Laukkanen
León
Lima
Liu
Moura
Seppänen
S

The most promising use of EIT is detection of breast cancer in combination with mammography

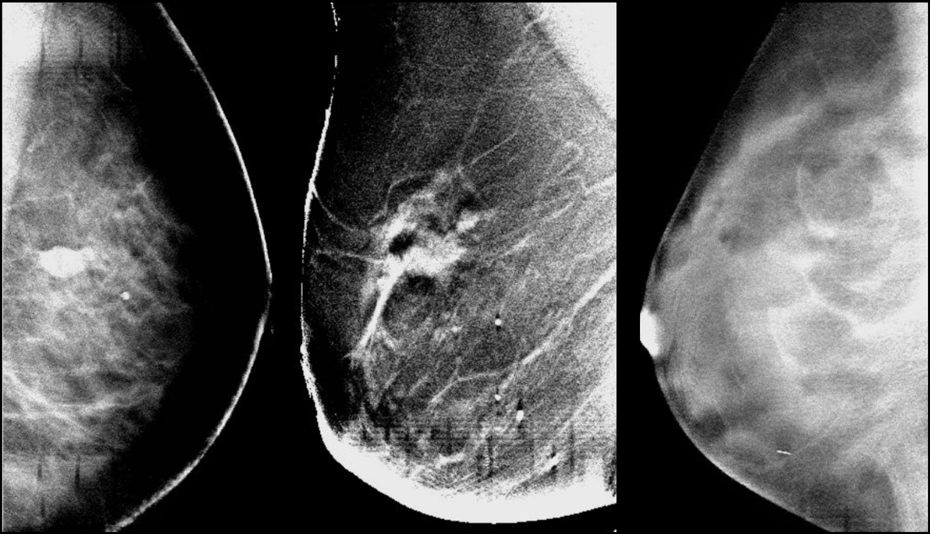
ACT4 and mammography devices

Radiolucent electrodes

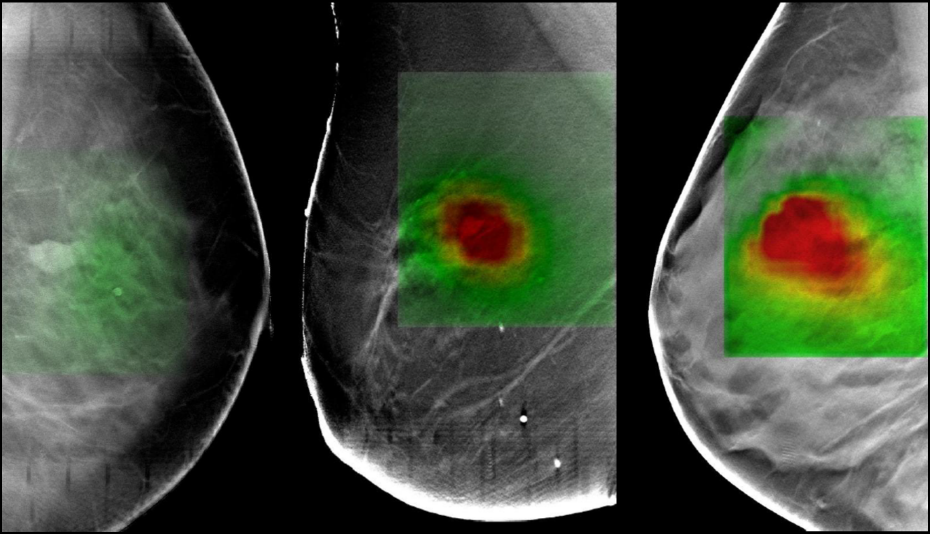


Cancerous tissue is up to four times more conductive than healthy breast tissue [Jossinet 1998]. The above experiment by **David Isaacson**'s team measures 3D X-ray mammograms and EIT data at the same time.

Which of these three breasts have cancer?

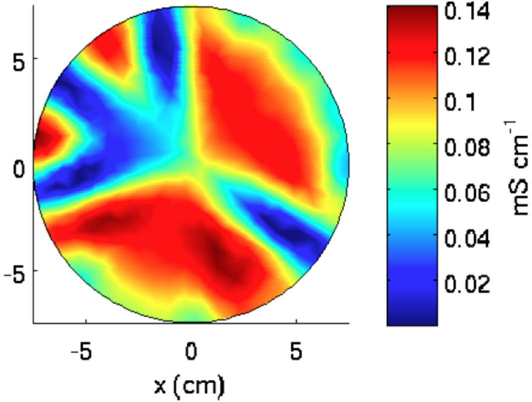


Spectral EIT can detect cancerous tissue



[Kim, Isaacson, Xia, Kao, Newell & Saulnier 2007]

EIT can be used for nondestructive testing: here for crack detection in concrete structures



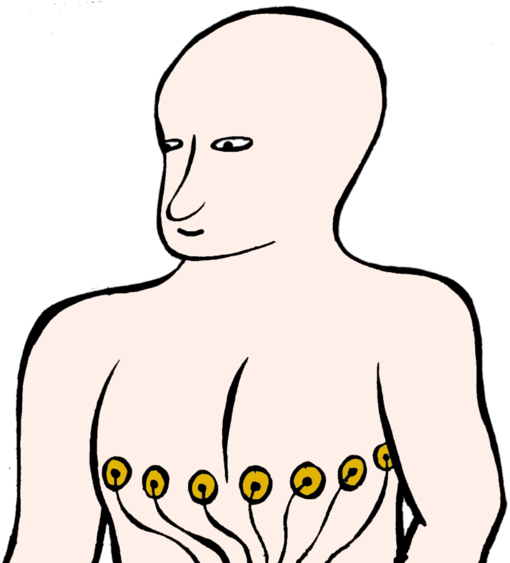
[Karhunen, Seppänen, Lehikoinen, Monteiro & Kaipio 2010]
[Karhunen, Seppänen, Lehikoinen, Monteiro, Kaipio, Blunt, Hyvönen]

Electrical impedance tomography (EIT) is an emerging medical imaging technique

Feed electric currents through electrodes. **Measure** the resulting voltages. Repeat the measurement for several current patterns.

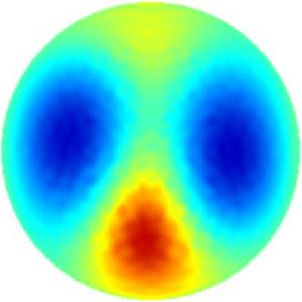
Reconstruct distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient's inner structure.

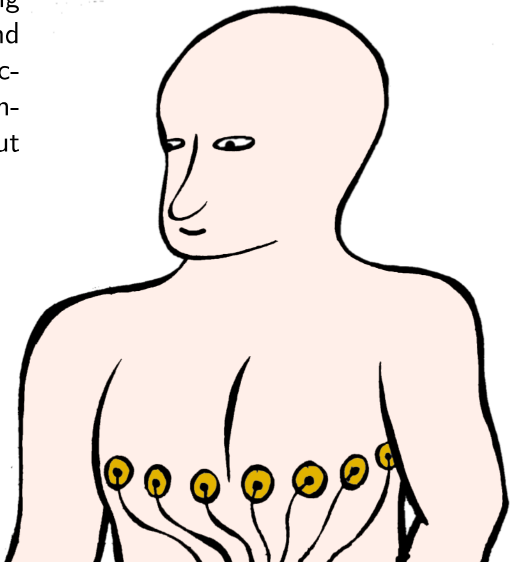
EIT is a harmless and painless imaging method suitable for long-term monitoring.



This talk concentrates on applications of EIT to chest imaging

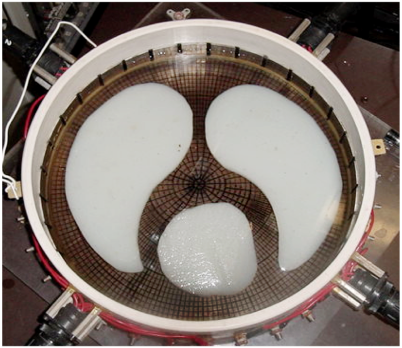
Medical applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.



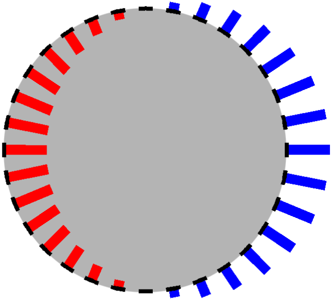


Note that EIT data collection involves applying several current patterns

Saline and agar phantom



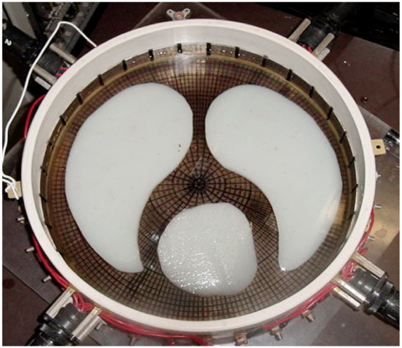
Apply current pattern $\cos\theta$



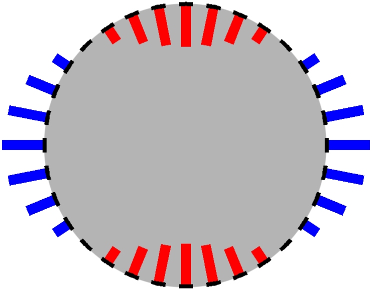
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



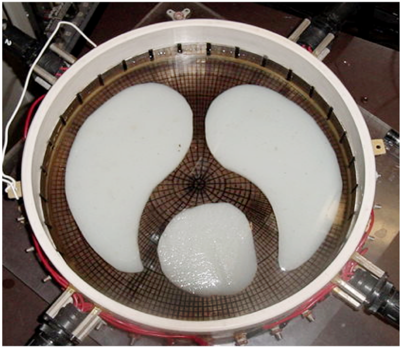
Apply current pattern $\cos 2\theta$



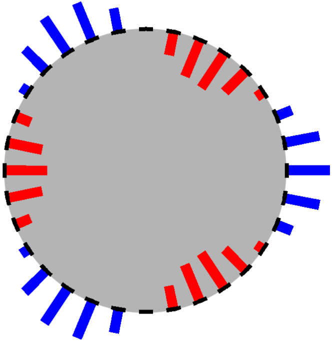
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



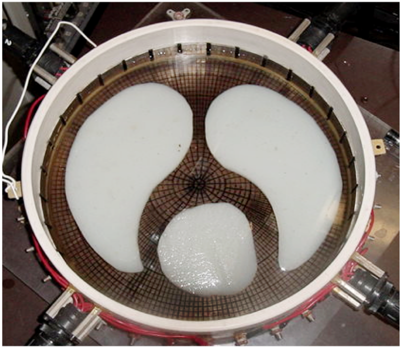
Apply current pattern $\cos 3\theta$



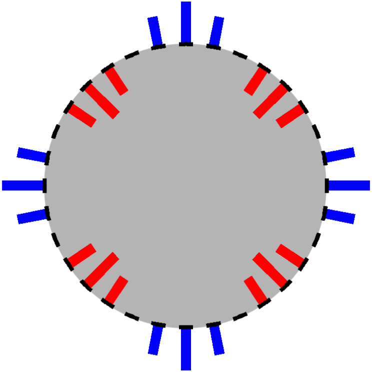
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



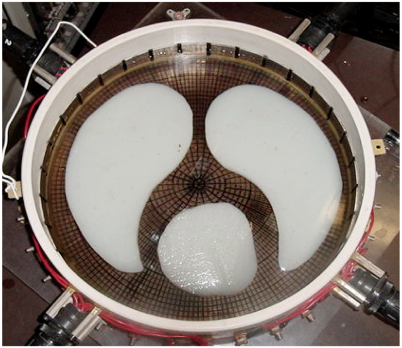
Apply current pattern $\cos 4\theta$



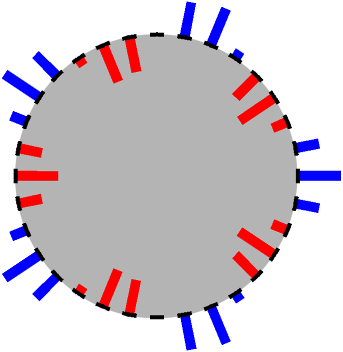
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



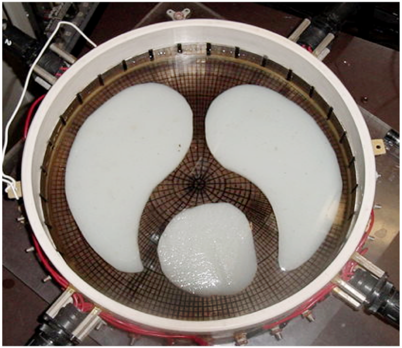
Apply current pattern $\cos 5\theta$



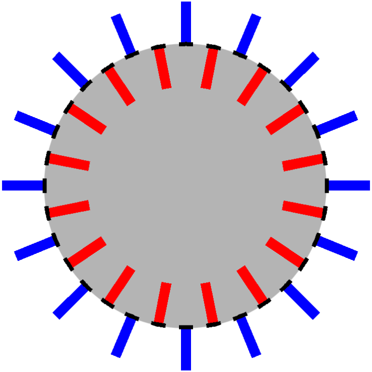
Measure the resulting voltages at the 32 electrodes

Note that EIT data collection involves applying several current patterns

Saline and agar phantom



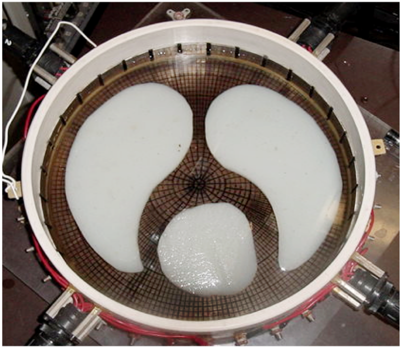
Apply current pattern $\cos 16\theta$



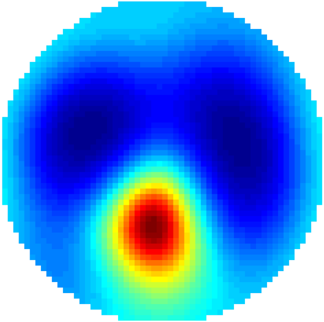
Measure the resulting voltages at the 32 electrodes

The D-bar method works for real EIT data, such as laboratory phantoms and *in vivo* human data

Saline and agar phantom



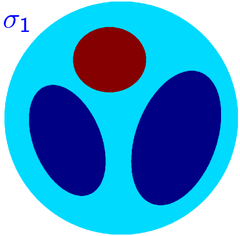
Reconstruction $(R = 4)$

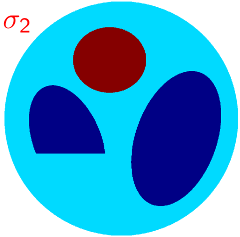


[Isaacson, Mueller, Newell & S 2004]
[Montoya 2012]

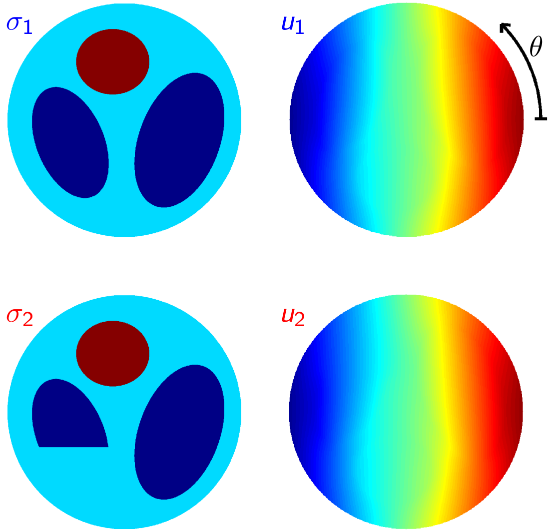
Outline

We illustrate the ill-posedness of EIT using a simulated example

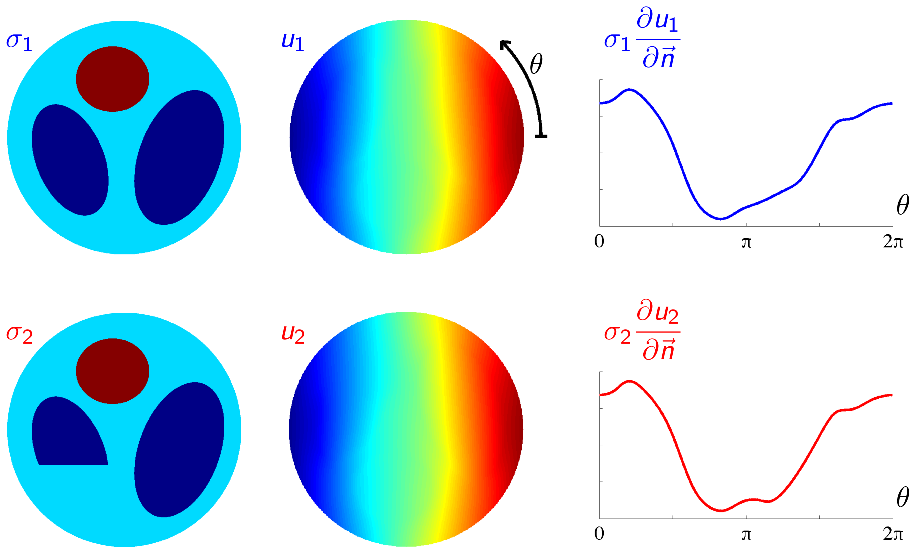




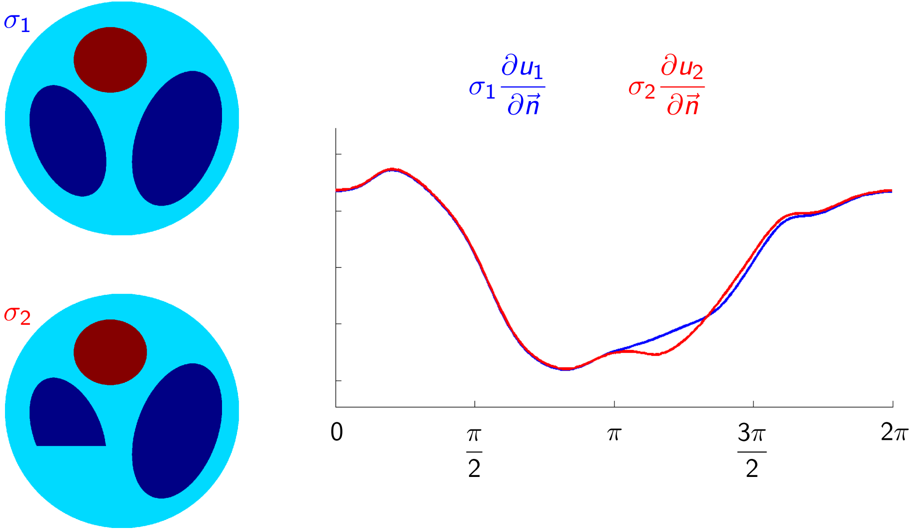
We apply the voltage distribution $f(\theta) = \cos\theta$ at the boundary of the two different phantoms



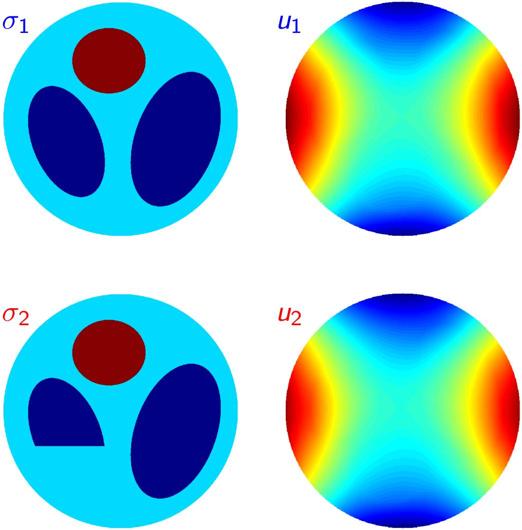
The measurement is the distribution of current through the boundary



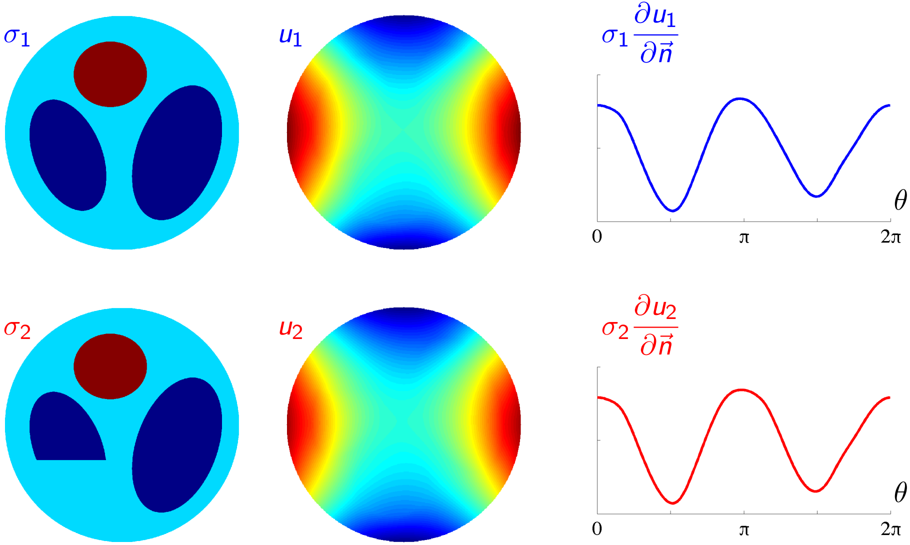
The current data are very similar, although the conductivities are quite different



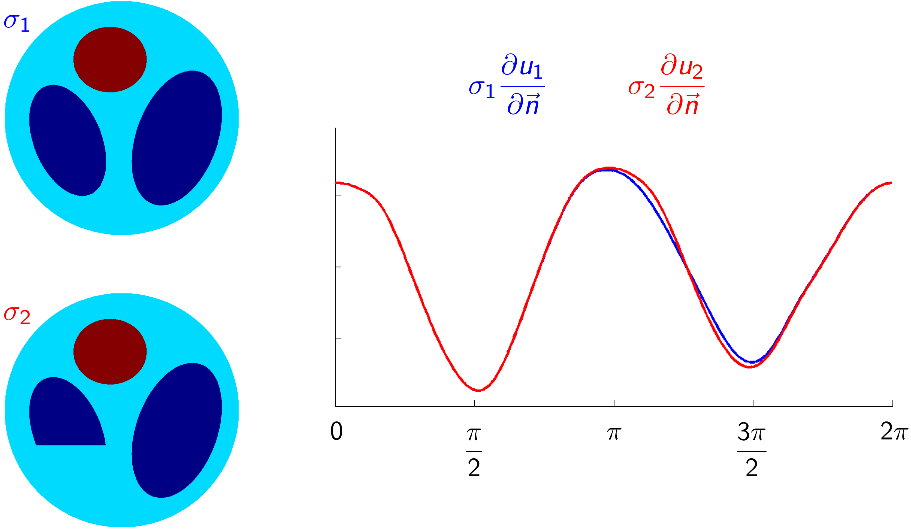
Let us apply the more oscillatory distribution $f(\theta) = \cos 2\theta$ of voltage at the boundary



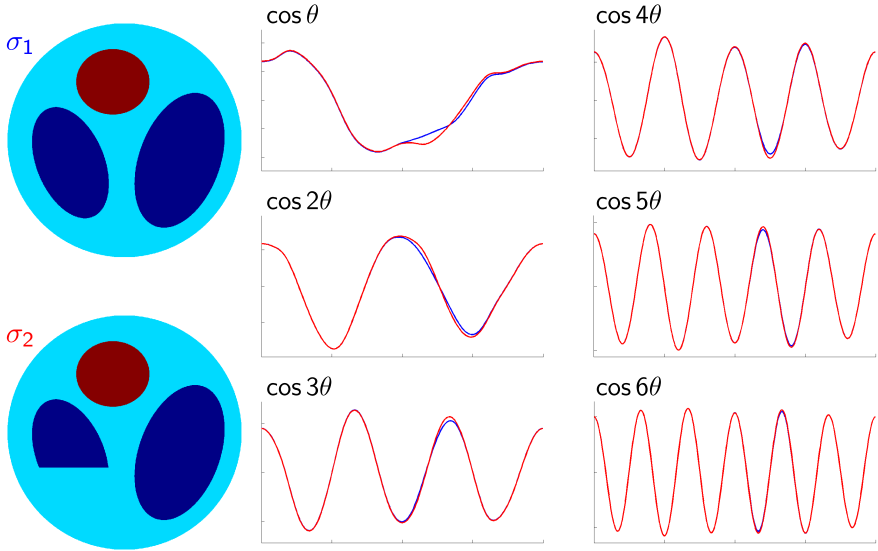
The measurement is again the distribution of current through the boundary



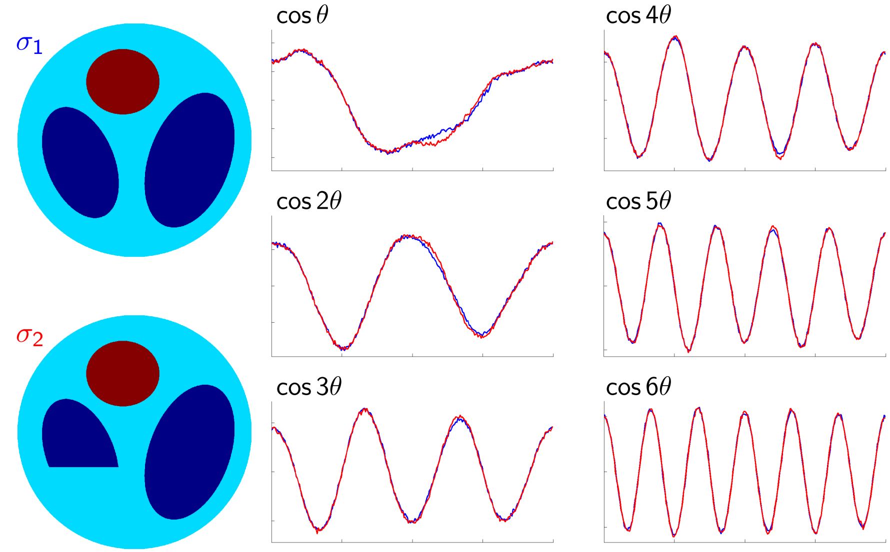
The current distribution measurements are almost the same



EIT is an ill-posed problem: big differences in conductivity cause only small effect in data

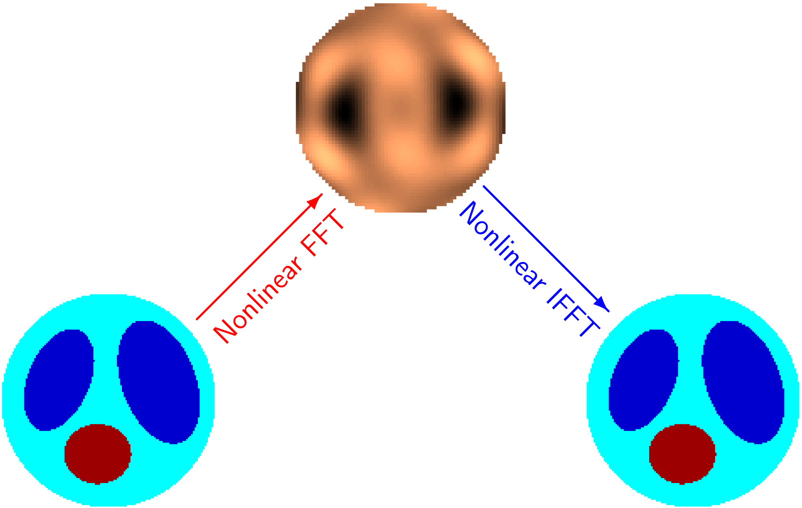


EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data

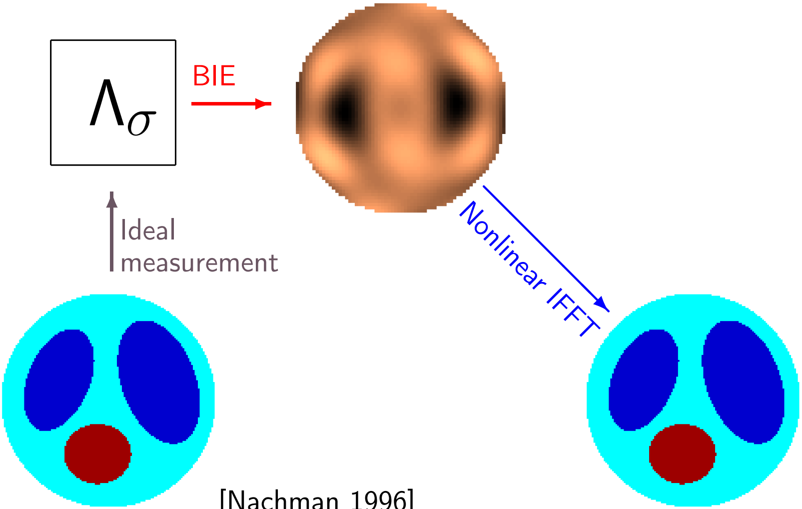


Outline

There exists a nonlinear Fourier transform adapted to electrical impedance tomography

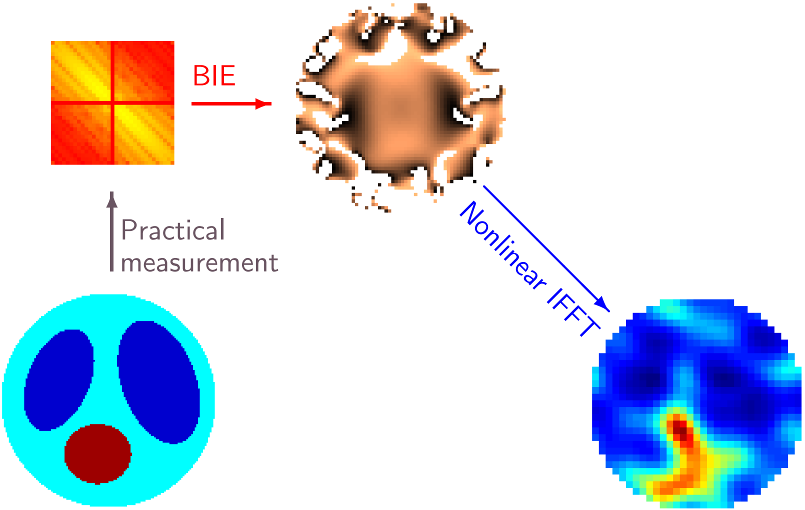


The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

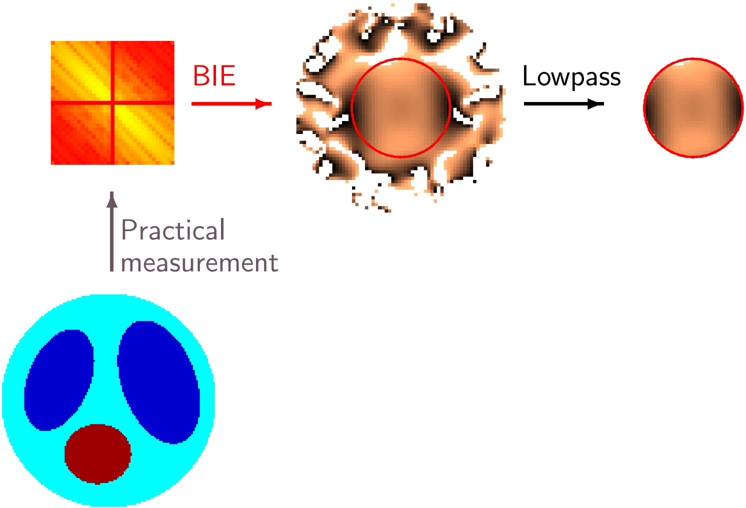


[Nachman 1996]

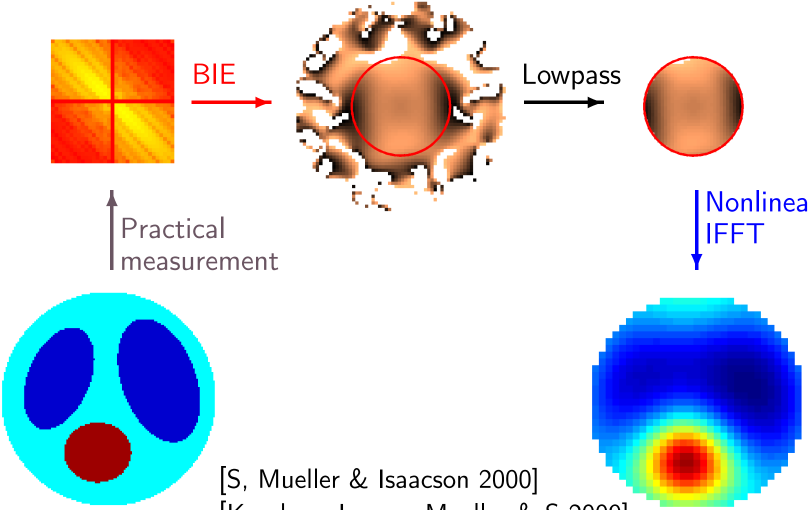
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies



We truncate away the bad part in the transform; this is a nonlinear low-pass filter



There is currently only one regularized method for reconstructing the full conductivity distribution



[S, Mueller & Isaacson 2000]
[Knudsen, Lassas, Mueller & S 2009]

D-bar reconstruction of *in vivo* chest data

(Loading DBarPerfMovie.avi)

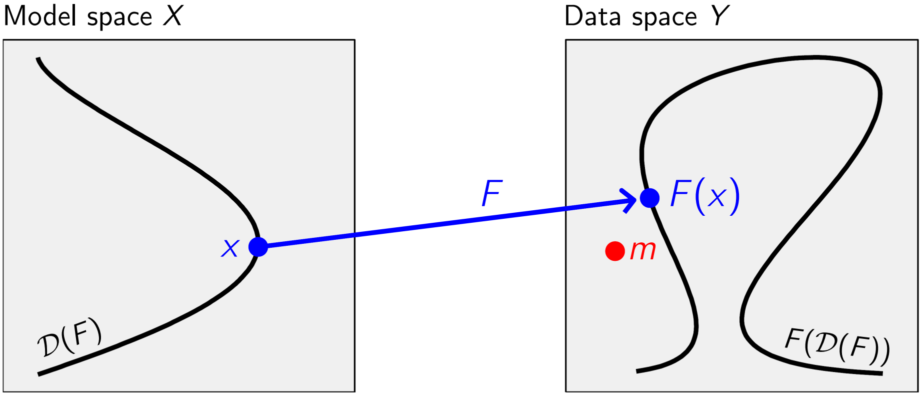
[Montoya & Mueller 2012]

Outline

Show other slides

Outline

Inverse problem = interpretation of an indirect measurement modelled by a forward map F



Consider the measurement model $m = F(x) + \varepsilon$. We want to know x, but all we can do is measure m that depends indirectly on x.

The practical measurement m can be thought of as infinite-precision data $F(x)$ corrupted with additive noise ε.

Ill-posed inverse problems are defined as opposites of well-posed direct problems

Hadamard: a problem is *well-posed* if the following conditions hold:

1. A solution exists,
2. The solution is unique,
3. The dependence of the solution on the input is continuous.

Well-posed direct problem: input x, find infinite-precision data $F(x)$.

Ill-posed inverse problem: input noisy data $m = F(x) + \varepsilon$, recover x.

The solution of an inverse problem is a *set of instructions* for recovering x stably from m

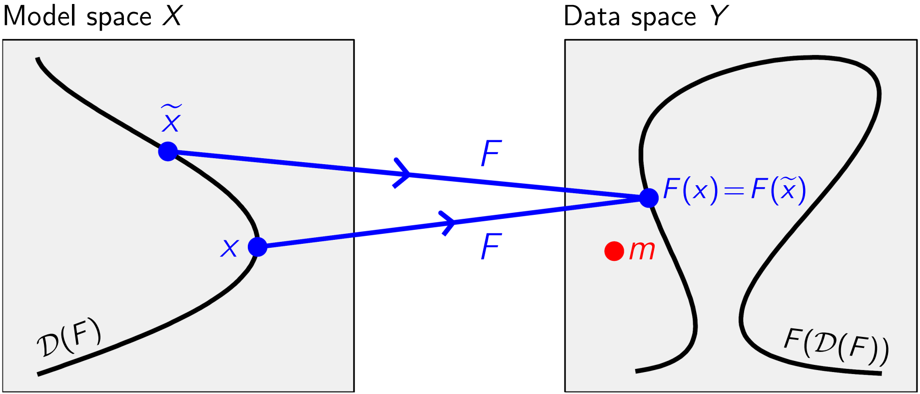
Those instructions need to be

(i) confirmed by rigorous mathematical analysis, and

(ii) implementable as an effective computational algorithm.

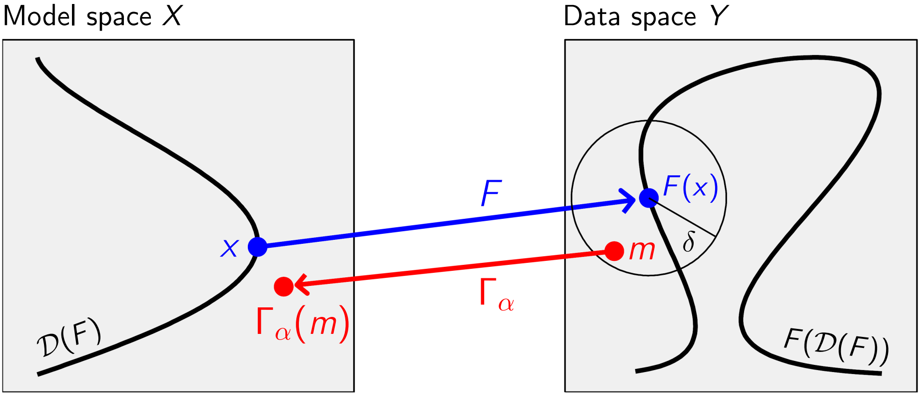
Since the forward map has no continuous inverse, it is impossible to recover x stably from m alone. The insufficient measurement data needs to be complemented by *a priori* knowledge.

Uniqueness: can two different objects produce the same infinite-precision data?



Ill-posedness means that the forward map F does not have a continuous inverse. Therefore, recovery of x from infinite-precision data $F(x)$ is unstable even if F is one-to-one. Furthermore, in general the data is not in the range: $m \notin F(\mathcal{D}(F))$.

Regularization means constructing a continuous map $\Gamma_\alpha : Y \to X$ that inverts F approximately



The reconstruction $\Gamma_{\alpha(\delta)}(m)$ needs to approach x along a continuous path as the noise level $\delta \to 0$.

The solution of an inverse problem is to design and implement the map Γ_α so that it contains appropriate prior information.

Goals of the course

1. Learn how to write a practical inverse problem in matrix form: $m = Af + \epsilon$
2. Learn how to detect ill-posedness from a matrix A using Singular Value Decomposition
3. Learn how to overcome ill-posedness by regularization
4. Acquire skills to solve practical inverse problems using Matlab
5. Learn to report your scientific findings in writing