Continuos deconvolution in 1D (formal approach)

Consider a function $f : \mathbb{R} \to \mathbb{R}$; assume $f \in L^1(\mathbb{R})$. Define Fourier transform of f by

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx .$$

Then it follows for the convolution

$$(g * f)(x) := \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$$

that

$$\widehat{(g*f)}(\xi) = \frac{1}{2\pi} \hat{g}(\xi)\,\hat{f}(\xi) .$$

Let the convolution kernel g be Gaussian:

$$g(x) = e^{-ax^2} \quad (a > 0).$$

Then $\hat{g}(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\xi^2/4a}$.

Now take the forward map to be $F(f) = g * f$. Is it injective? Assume that

$$F(f) = F(h).$$

Then

$$\begin{aligned} 0 &= F(f) - F(h) \\ &= g*f - g*h \\ &= g*(f-h). \end{aligned}$$

Fourier transform to get

$$0 = \sqrt{\frac{\pi}{a}}\, e^{-\xi^2/4a}\, \widehat{(f-h)} \Rightarrow f - h \equiv 0$$

It can be shown that the Fourier transform is continuous in $L^2(\mathbb{R}^n)$:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx$$

$$\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

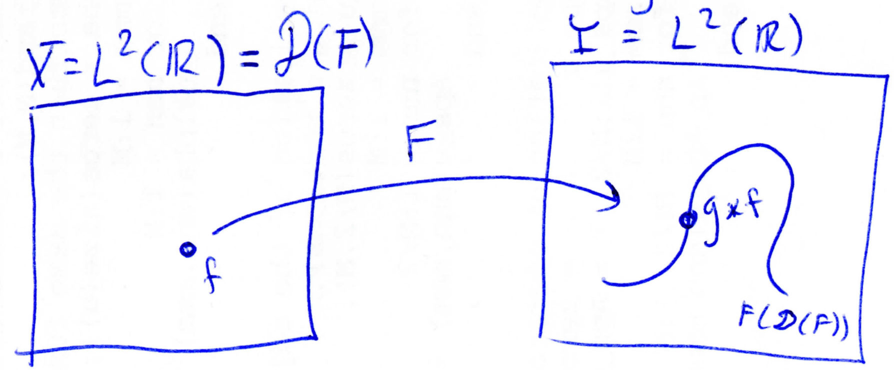
Also, it has an inverse transform

$$\mathcal{F}^{-1}(\hat{f})(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi .$$

Note Parseval's identity:

$$\| f \|_2 = c \| \hat{f} \|_{L^2} .$$

Consider the diagram



Now $F(\mathcal{D}(F)) \subseteq Y$ is a proper subset since $g*f$ is a smooth function. Actually, $g*$ is a compact operator.

Question: is the inverse problem of recovering f from $m = g*f + \varepsilon$ well-posed or ill-posed?

Clearly, $g*f+\varepsilon$ is not necessarily in $F(\mathcal{D}(F))$ as ε may be nonsmooth.

Furthermore, consider the formula $\hat{m} = \widehat{g*f} + \hat{\varepsilon}$ and the naive inversion

$$\sqrt{\tfrac{a}{\pi}}\, e^{\xi^2/4a}\, \hat{m}(\xi) = \hat{f}(\xi) + \sqrt{\tfrac{a}{\pi}}\, e^{\xi^2/4a}\, \hat{\varepsilon}(\xi).$$

Here the error $\mathcal{F}^{-1}\left(\sqrt{\tfrac{a}{\pi}}\, e^{\xi^2/4a}\, \hat{\varepsilon}(\xi)\right)$ can be huge even if $\|\varepsilon\|_2$ is small. The essential point is where the support of $\hat{\varepsilon}(\xi)$ is located in the frequency domain.

Typical measurement noise has significant high-frequency content.