Inverse problems course, spring 2014 Exercise 1 solutions (January 21-24, 2014)
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Theoretical exercises:

T1. Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{for } 0.4 \leq x \leq 0.6, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the function $g * g$ analytically (by hand), where

$$(g * g)(x) = \int_{-\infty}^{\infty} g(x')g(x - x')\, dx'.$$

Outside which interval $[a, b] \subset \mathbb{R}$ is $(g * g)(x) = 0$?

Answer

Using the definition of convolution and function g we get

$$(g * g)(y) = \int_{-\infty}^{\infty} g(x)g(y - x)\, dx = \int_{0.4}^{0.6} g(y - x)\, dx.$$

If we do a substitution $x \mapsto -x + y$ we get

$$(g * g)(y) = -\int_{-0.4+y}^{-0.6+y} g(x)\, dx = \int_{-0.6+y}^{-0.4+y} g(x)\, dx.$$

From this formula we can deduce that, if $y < 0.8$ or $y > 1.2$ then $(g*g)(y) = 0$. So $(g * g)(y) = 0$ outside interval $[0.8, 1.2]$.

If $y \in [0.8, 1.0]$, we get that

$$(g * g)(y) = \int_{-0.6+y}^{-0.4+y} g(x)\, dx = \int_{0.4}^{-0.4+y} g(x)\, dx = -0.4 + y - 0.4 = y - 0.8.$$

If $y \in [1.0, 1.2]$, we get that

$$(g * g)(y) = \int_{-0.6+y}^{-0.4+y} g(x)\, dx = \int_{-0.6+y}^{0.6} g(x)\, dx = 1.2 - y.$$

Putting everything together we have shown that

$$(g * g)(y) = \begin{cases} 0,\ y < 0.8 \\ y - 0.8,\ y \in [0.8, 1.0] \\ 1.2 - y,\ y \in [1.0, 1.2] \\ 0,\ y > 1.2 \end{cases}$$

T2. Let the discrete point spread function $p \in \mathbb{R}^5$ and the vector $f \in \mathbb{R}^{10}$ be defined by

$$\begin{aligned} \widetilde{p} &= [\widetilde{p}_{-2}, \widetilde{p}_{-1}, \widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2]^T = [1,1,1,1,1]^T, \\ f &= [f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}]^T = [0,0,0,0,1,1,0,0,0,0]^T. \end{aligned}$$

Compute the discrete convolution vector $(\widetilde{p} * f) \in \mathbb{R}^{10}$ by

$$(\widetilde{p} * f)_j = \sum_{\ell=-2}^{2} \widetilde{p}_\ell f_{j-\ell},$$

where $f_{j-\ell}$ is defined using periodic boundary conditions for the cases $j - \ell < 1$ and $j - \ell > n$.

Answer

Let's calculate the first one in detail.

$$(\widetilde{p} * f)_1 = \sum_{\ell=-2}^{2} \widetilde{p}_\ell f_{1-\ell} = \widetilde{p}_{-2} f_3 + \widetilde{p}_{-1} f_2 + \widetilde{p}_0 f_1 + \widetilde{p}_1 f_0 + \widetilde{p}_2 f_{-1}.$$

Due the periodic boundary condition we have that $f_0 = f_{10}$ and $f_{-1} = f_9$. Making these substitutions we get

$$(\widetilde{p} * f)_1 = \widetilde{p}_{-2} f_3 + \widetilde{p}_{-1} f_2 + \widetilde{p}_0 f_1 + \widetilde{p}_1 f_{10} + \widetilde{p}_2 f_9 = 0 + 0 + 0 + 0 + 0 = 0.$$

Next we give a list of correct answers.

$$\begin{pmatrix} (\widetilde{p} * f)_1 = 0 & (\widetilde{p} * f)_2 = 0 & (\widetilde{p} * f)_3 = 1 & (\widetilde{p} * f)_4 = 2 & (\widetilde{p} * f)_5 = 2 \\ (\widetilde{p} * f)_6 = 2 & (\widetilde{p} * f)_7 = 2 & (\widetilde{p} * f)_8 = 1 & (\widetilde{p} * f)_9 = 0 & (\widetilde{p} * f)_{10} = 0 \end{pmatrix}$$

T3. Take $\Delta x = \frac{1}{10}$ and compute the normalized point spread function

$$p = \left(\Delta x \sum_{j=-2}^{2} \widetilde{p}_j \right)^{-1} \widetilde{p}.$$

Compute the discrete convolution vector $(\widetilde{p} * f) \in \mathbb{R}^{10}$ with vector $f \in \mathbb{R}^{10}$ as in exercise T2 except that $f_1 = 2$. Be careful with the periodic boundary condition!

Answer

$$p = \left(\Delta x \sum_{j=-2}^{2} \widetilde{p}_j \right)^{-1} \widetilde{p} = \frac{1}{\frac{1}{10} 5} (1,1,1,1,1)^T = (2,2,2,2,2)^T.$$

In this case function f look's like

$$f = [2,0,0,0,1,1,0,0,0,0]^T.$$

Let's calculate number $(\widetilde{p} * f)_1$ in detail. Using the vectors given for this problem and remembering the periodic boundary condition we get

$$(\widetilde{p} * f)_1 = \widetilde{p}_{-2} f_3 + \widetilde{p}_{-1} f_2 + \widetilde{p}_0 f_1 + \widetilde{p}_1 f_0 + \widetilde{p}_2 f_{-1} = 0 + 0 + 2 + 0 + 0 = 2.$$

Next we give a list of correct answers.

$$\begin{pmatrix} (\widetilde{p} * f)_1 = 2 & (\widetilde{p} * f)_2 = 2 & (\widetilde{p} * f)_3 = 3 & (\widetilde{p} * f)_4 = 2 & (\widetilde{p} * f)_5 = 2 \\ (\widetilde{p} * f)_6 = 2 & (\widetilde{p} * f)_7 = 2 & (\widetilde{p} * f)_8 = 1 & (\widetilde{p} * f)_9 = 2 & (\widetilde{p} * f)_{10} = 2 \end{pmatrix}$$

Matlab exercises:

M1. Download the following files from the course webpage:
`DC_PSF.m`
`DC_target.m`
`DC_convmtx.m`
`DC1_cont_data_comp.m`
`DC1_cont_data_plot.m`
`DC2_discretedata_comp.m`
`DC2_discretedata_plot.m`

(a) Repeat the experiment done at the lecture: choose `n = 32`, `n = 64`, `n = 128` and `n = 256` in line 12 of the file `DC2_discretedata_comp.m` and observe how the approximation error becomes smaller. (In other words, the blue dots in the image entitled *Data with inverse crime* get closer to the red function as `n` grows.)

(b) Now choose a to be smaller than 0.04 in line 10 of file `DC1_cont_data_comp.m` and run it. Repeat the experiment in (i). Is the convergence of blue dots to the red function slower or faster, especially near the discontinuities of the original signal? Why is this?

(c) Now choose a to be greater than 0.04 in line 10 of file `DC1_cont_data_comp.m` and run it. Repeat the experiment in (i). Is the convergence of blue dots to the red function slower or faster? Why?

Answer

Choosing $a = \{0.04, 0.035, 0.03, 0.025\}$ and $n = \{32, 64, 128, 256\}$ one can see a difference especially with $n = \{32, 64\}$. It seems that the blue dots converge to red line slower if a get's smaller.

Choosing $a = \{0.04, 0.045, 0.05, 0.055\}$ and $n = \{32, 64, 128, 256\}$ one can see a difference especially with $n = \{32, 64\}$. It seems that the blue dots converge to red line faster if a get's larger.

The red function is greated analytically by formula

$$(\psi_a * f)(y) = \int_{-\infty}^{\infty} \psi_a(x) f(x - y) dx, \tag{1}$$

where

$$\psi_a(x) = \begin{cases} C_a(x-a)^2(x+a)^2, \ |x| \le a \\ 0, \ |x| > a \end{cases} \tag{2}$$

is SPF. If parameter a get's smaller, we can see from definition of SPF that the support of PSF goes smaller. It is shown in a basic course of Real analysis that

$$(\psi_a * f) \in C^1(\mathbb{R}) \text{ and } |f - (\psi_a * f)|_{L^1} \longrightarrow 0 \text{ if } a \longrightarrow 0.$$

In other words this means that $(\psi_a * f)$ approximates function f better and better as a get's smaller.

If we now consider the discrete setting, where variable n is the number of points in which we calculate f. Considering the definitions made in the textbook we see that Discrete point spread function, which is matrix (2.14) in textbook, has more zero elements for fixed n if a decreases. Let $\mathbf{f}(n)$ be the vector of formula (2.7) and $\mathbf{P}(a,n)$ be the point spread matrix with respect to parameters a and n. The blue dots are the vector

$$\frac{1}{n}\mathbf{P}(a,n)\mathbf{f}(n). \tag{3}$$

If we pick one entry, say the j^{th} one, of vector in line (3) we have formula

$$\frac{1}{n}\sum_{\ell=1}^{n} p(a)_{\ell,j} f_\ell. \tag{4}$$

For a big enough we see, that the sum (4) approximates the Riemannian sum

$$\frac{1}{n}\sum_{\ell=1}^{n} \psi_a(x_\ell) f(x_\ell - x_j).$$

better. This holds due the construction of $\mathbf{P}(a,n)$ and since we have a crid where distance between neighboring points is constant $\Delta x = 1/n$. It is shown in the basic course of analysis that next formula holds.

$$(\psi_a * f)(x_j) = \int_{-\infty}^{\infty} \psi_a(x) f(x - x_j) dx \sim \frac{1}{n}\sum_{\ell=1}^{n} \psi_a(x_\ell) f(x_\ell - x_j). \tag{5}$$

There fore for a bigger parameter a the blue dots approximate the red function better if we keep the parameter n intact.

On second thought there is a critical point where the situation above doesn't work anymore. Suppose that $a < \Delta x = \frac{1}{n}$. Then it holds that $\nu = 0$ and $\widetilde{\mathbf{P}}(a,n) = \mathrm{diag}(\psi_a(0), \ldots, \psi_a(0))$, where $\psi_a(0) = C_a a^4$. Our Matlab code does not how ever use $\widetilde{\mathbf{P}}(a,n)$, but the normalized $\mathbf{P}(a,n) = \mathrm{diag}(p, \ldots, p)$, where

$$p = (\Delta x \widetilde{p})^{-1} \widetilde{p} = (\Delta x)^{-1}.$$

If we now consider vector (3) in our situation we note that

$$\frac{1}{n}\mathbf{P}(a,n)\mathbf{f}(n) = \Delta x \operatorname{diag}((\Delta x)^{-1}, \ldots, (\Delta x)^{-1})\mathbf{f}(n) = \mathbf{f}(n).$$

There fore we get perfect results.