Inverse problems course exam — Published on February 20, 2014
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Please return your hand-written solutions to Problems 1–2 in the examination session on Friday, March 14, 2014, at 12:15. The hall is Exactum C123.

Here are detailed instructions concerning Problem 3. Download the LaTeX file `Laplacereport.tex` from the course website. Report your answers to sub-problems 3(a–c) below using that file as a starting point. Feel free to change the titles or to add subsections as needed. Aim for a report of not more than 10 pages. Please include plots of reconstructions, singular values, and other relevant numerical data. Please return the printed pdf report about Problem 3 in the examination session mentioned above. Also, please send your Matlab codes by email to me (`Samuli.Siltanen@helsinki.fi`) at latest on Thursday, March 13.

If you have practical questions about how to complete this home exam, please contact me by email before February 27, 2014.

1. A problem of the form *find f, when $m = Af + \epsilon$ is given* is called an inverse problem if it is *not* well-posed in the sense of Hadamard.

    (a) Formulate Hadamard's definition of a well-posed problem in the context of the above inverse problem.

    (b) Write down the definition of a *regularization strategy*. Draw a picture showing the data space, the parameter space, the direct map, m, f and the regularized inverse. Explain how the regularized solution can be understood as a noise-robust approximation of f.

    (c) Give an example of a practical ill-posed inverse problem (other that deconvolution, X-ray tomography or inversion of Laplace transform) where Hadamard's continuity condition fails. Explain why the forward map does not have a continuous inverse in your example. Include references to the literature or Internet resources you used.

2. Download the files
`XRsparseA_NoCrimeData_comp.m`,
`XRsparseD_aTV_comp.m`, `XRsparseD_aTV_plot.m`,
`XR_misfit.m`, `XR_misfit_grad.m`
`XR_aTV_feval.m`, `XR_aTV_fgrad.m`,
`XR_aTV.m`, `XR_aTV_grad.m`,
from the course website. They solve the 512×512 sparse-angle X-ray tomography problem using total variation regularization with a smooth approximation to the absolute value function. See Sections 6.4 and 9.1 of the textbook.

   (a) Explain mathematically how the Barzilai-Borwein minimization method is used for computing the regularized solution. (No proofs are needed, just the description of the solution method in your own words.)

   (b) Describe the purpose of each of the above Matlab files. In other words, explain the role of each Matlab file in the mathematical method you discussed in (a).

   (c) Download the files
   `XRsparseC_Tikhonov_comp.m`,
   `XRsparseC_Tikhonov_plot.m`,
   from the course website. Experiment with different values of the regularization parameter for both TV regularization and Tikhonov regularization. What is the lowest relative error in the reconstruction you can achieve using each of the two methods? Please report the parameter values that resulted in the lowest errors.

3. Let $f : [0, \infty) \to \mathbb{R}$. The Laplace transform F of f is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt, \quad s \in \mathbb{C}, \tag{1}$$

provided that the integral converges. The direct problem is to determine F for a given function f according to (1). The inverse problem is: *given a Laplace transform F, find the corresponding function f.*

Assume we know the values of F at these real-valued points:

$$0 < s_1 < s_2 < \ldots < s_n < \infty.$$

Then we may approximate the integral in (1) for example with the trapezoidal rule as

$$\begin{aligned}\int_0^\infty e^{-st} f(t) dt \approx \frac{t_k}{k} \bigg(& \frac{1}{2} e^{-st_1} f(t_1) + e^{-st_2} f(t_2) + e^{-st_3} f(t_3) + \ldots \\ & + e^{-st_{k-1}} f(t_{k-1}) + \frac{1}{2} e^{-st_k} f(t_k) \bigg),\end{aligned} \tag{2}$$

where vector $t = [t_1 \; t_2 \; \ldots \; t_k]^T \in \mathbb{R}^k$, $0 \le t_1 < t_2 < \ldots < t_k$, contains the points at which the unknown function f will be evaluated. By denoting $f_\ell = f(t_\ell)$, $\ell = 1, \ldots, k$, and $m_j = F(s_j)$, $j = 1, \ldots, n$, and using (2), we get a linear model of the form $m = Af + \epsilon$ with

$$A = \frac{t_k}{k} \begin{bmatrix} \frac{1}{2} e^{-s_1 t_1} & e^{-s_1 t_2} & e^{-s_1 t_3} & \ldots & e^{-s_1 t_{k-1}} & \frac{1}{2} e^{-s_1 t_k} \\ \frac{1}{2} e^{-s_2 t_1} & e^{-s_2 t_2} & e^{-s_2 t_3} & \ldots & e^{-s_2 t_{k-1}} & \frac{1}{2} e^{-s_2 t_k} \\ \vdots & & & & & \vdots \\ \frac{1}{2} e^{-s_n t_1} & e^{-s_n t_2} & e^{-s_n t_3} & \ldots & e^{-s_n t_{k-1}} & \frac{1}{2} e^{-s_n t_k} \end{bmatrix}. \tag{3}$$

(a) Compute numerically and plot the Laplace transform of

$$f(t) = \begin{cases} 1, & \text{for } 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Construct matrix A given by (3) for a suitable choice of points t_ℓ and s_j. Compute the singular values of A. Do you detect ill-posedness?

(c) Use truncated SVD to compute the inverse Laplace transform of f.