FUNKTIONAALIANALYYSI II, 2014
EXERCISES, SET 3
TO BE RETURNED ON MON. APRIL 14th AT LATEST, PERSONALLY OR TO THE MAILBOX OF J.T.

1. Given $f \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$, show that $f * T \in C^\infty(\mathbb{R}^n)$ (cf. Theorem 4.20 of the lecture notes).

2. Prove that, if $\varepsilon \to 0$, then $J_\varepsilon \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$, where J_ε is the mollifier of Chapter 5 of the lecture notes. Given $T \in \mathcal{S}'(\mathbb{R}^n)$, show also that $J_\varepsilon * T \to T$ in $\mathcal{S}'(\mathbb{R}^n)$, if $\varepsilon \to 0$ and $\mathcal{S}'(\mathbb{R}^n)$ is endowed with the weak topology. This amounts to showing that

$$\langle \varphi, J_\varepsilon * T \rangle \to \langle \varphi, T \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

3. Let $\Omega :=]0, \infty[\subset \mathbb{R}$. Show that the following inclusions are *strict*, i.e., the spaces are not the same: $C^1(\bar{\Omega}) \subset C^1_B(\Omega) \subset C^1(\Omega)$.

4. For all λ, $0 < \lambda < 1$, construct a function $f : \mathbb{R} \to \mathbb{R}$, which is an element of the Hölder–space $C^{0,\lambda}(\bar{\mathbb{R}})$ but not that of $C^1(\bar{\mathbb{R}})$.

5. Let $a > 0$ and $b > 0$, and $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and

$$f(x) := x^{-a} \quad , \quad g(x) := x^{-b}$$

for $|x| \leq 1$, and $f(x) = g(x) = 0$ for $|x| > 1$.
What does the Young inequality imply about the existence of the function $f * g$ as an element in the space $L^r(\mathbb{R})$ for different a, b and $1 \leq r \leq \infty$?

6. Show that the operator P defined just before Theorem 5.12 in the lecture notes is an isometry from the Sobolev space onto a closed subspace of X.

7. Show by an example, that $W_0^{1,1}(\Omega) \neq W^{1,1}(\Omega)$ for $\Omega := \{|x| < 1\} \subset \mathbb{R}^2$.

8. Using the definition, show that the open unit ball of $\mathbb{R}^2$ has the cone property.

9. For which values of $a \in \mathbb{R}$ does the function $|x|^a = (x_1^2 + x_2^2)^{a/2}$ belong to the Sobolev space $W^{1,p}(\omega)$, where ω is the open unit ball $\{|x| < 1\}$ of $\mathbb{R}^2$ and $1 \leq p \leq \infty$? Also, the same for the function $|x|^a = (x_1^2 + x_2^2 + x_3^2)^{a/2}$, the Sobolev space $W^{1,p}(\Omega)$, and Ω as the open unit ball of $\mathbb{R}^3$.