FUNKTIONAALIANALYYSI II, 2014
EXERCISES, SET 2

1. Let us consider here *simple functions of intervals* χ in $\mathbb{R}$, which are constants on some subintervals of $\mathbb{R}$, e.g. $\chi(x) = 2$ for $1 \leq x \leq 4$ and $\chi(x) = -1$ for $4 < x \leq 5$ and $\chi(x) = 0$ elsewhere.
Find a sequence of simple functions which converges (weakly) in $\mathcal{D}'(\mathbb{R})$ to the distribution a) δ_0, b) δ_0'.
Is it true that in $\mathcal{D}'(\mathbb{R})$,

$$\lim_{n\to\infty} n\delta_{-1/n} - n\delta_{1/n} = 2\delta_0'.$$

2. Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences of, say, functions in $L^1(\mathbb{R})$ such that $f_n \to \delta_a$, $g_n \to \delta_b$ in $\mathcal{D}'(\mathbb{R})$, where a, $b \in \mathbb{R}$ are fixed. Let us assume in addition that the $L^1(\mathbb{R})$-norms of all functions are bounded by some constant $C > 0$. Prove that

$$\lim_{n\to\infty} f_n \otimes g_n \to \delta_{(a,b)}$$

in $\mathcal{D}'(\mathbb{R}^2)$.

3. Give examples of sequences of simple functions of *rectangles* and *discs* of $\mathbb{R}^2$, which converge to $\delta_{\bar{0}}$ in $\mathcal{D}'(\mathbb{R}^2)$. Problem 2 may be of some use.

4. Write the distribution $T \in \mathcal{D}'(\mathbb{R})$,

$$T := \delta_5 - \frac{d\delta_0}{dx}$$

as a derivative of a continuous function on $\mathbb{R}$.

5. Let $Y : \mathbb{R} \to \mathbb{C}$ be the step function. Show that $d\delta_0/dx * Y = \delta_0$ and that $1 * d\delta_0/dx = 0$. Calculate

$$1 * \left(\frac{d\delta_0}{dx} * Y\right) \quad \text{and} \quad \left(1 * \frac{d\delta_0}{dx}\right) * Y. \tag{0.1}$$

This seems to violate the associative law. What's wrong?

6. As regards to Theorem 3.11 of the lecture notes, show the following facts for all test functions $\varphi \in \mathcal{D}(\Omega \times \Omega')$:
a) Given a fixed $y \in \Omega'$, the function $x \mapsto \varphi(x, y)$ is an element of $\mathcal{D}(\Omega)$.
b) Given $T \in \mathcal{D}'(\Omega')$, the function $x \mapsto \langle \varphi(x, y), T_y \rangle$ is at least differentiable in x. (It is actually C^∞, can you prove it?)

7. Given the domains Ω and Ω', one can also define the tensor product space $C^\infty(\Omega) \otimes C^\infty(\Omega')$ in the same way as in the Definition 3.9 of the lecture notes, in particular defining $f \otimes g$ as the pointwise product (3.19) for $f \in C^\infty(\Omega)$, $g \in C^\infty(\Omega')$. It is then quite trivial that $C^\infty(\Omega) \otimes C^\infty(\Omega') \subset C^\infty(\Omega \times \Omega')$.
Formulate a result analogous to Theorem 3.10 for distributions with compact support.

8. Assume $\Omega = \Omega' = \mathbb{R}$ in the situation of Ex. 7.
a) Does the function $\sin(x + y)$, $x, y \in \mathbb{R}$, belong to $C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R})$ (trigonometric formulas!)

b) Can you approximate the function $e^{-(x+y)^2}$ by elements of $C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R})$? (If it is too difficult to consider sup-norms of all derivatives, prove the approximation at least for the sup-norm of the function itself.)

9. Let $f \in C^\infty(\mathbb{R}^n)$, and let $\gamma_{k,m}$ be as in (4.1) of the lecture notes. Show that the condition "$\gamma_{k,m}(f) < \infty$ for all k and m" is equivalent to the condition

$$\text{"} \lim_{|x|\to\infty} |x|^k |D^\alpha f(x)| = 0 \tag{0.2}$$

for all k and α".