FUNKTIONAALIANALYYSI II, 2014
EXERCISES, SET 1

1. Prove that the space $\mathcal{D}(\mathbb{R})$ is not sequentially complete when endowed with the topology of $C^\infty(\mathbb{R})$.

2. Let us denote $C_c(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \,:\, f \text{ is continuous and has compact support}\}$. Given a continuous, increasing function $v : [0, \infty[\to]0, \infty[$ (such functions are called weights), define the weighted norm

$$\|f\|_v := \sup_{x \in \mathbb{R}} v(|x|)|f(x)| \tag{0.1}$$

for all $f \in C_c(\mathbb{R})$. Information: A most natural way to topologize $C_c(\mathbb{R})$ is to consider it as the intersection, or "projective limit" of the spaces $C_v(\mathbb{R})$, where v runs through all weights as described above and $C_v(\mathbb{R})$ consists of continuous functions f such that the norm (0.1) is well defined (=finite supremum). In other words, the topology of $C_c(\mathbb{R})$ is determined by the family of all norms (0.1).

a) Show that a continuous $f : \mathbb{R} \to \mathbb{C}$ belongs to $C_c(\mathbb{R})$, if and only if the expression (0.1) is finite for *all* weights v.

b) Given an arbitrary sequence $(v_k)_{k=1}^\infty$ of weights, prove that there exists a weight w as follows: for all k there exists a constant $C_k > 0$ such that

$$w(x) \geq C_k v_k(x) \quad \text{for all } x \in \mathbb{R}. \tag{0.2}$$

Some more information: This last phenomenon is the philosophical reason for the fact that the space $C_c(\mathbb{R})$ (and in the same way also the space $\mathcal{D}(\mathbb{R})$) are not metrizable, i.e., its topology cannot be gotten from a countable number of (semi)norms. For example, given an arbitrary sequence $(v_k)_{k=1}^\infty$ of weights, one can always find a continuous function $g : \mathbb{R} \to \mathbb{C}$ with $\text{supp}\,(g) = \mathbb{R}$, which belongs to all of the spaces $C_{v_n}(\mathbb{R})$.

3. Show that the linear mapping T is continuous from the space $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$, if a) $Tf(x) := f(x+3)$, b) $Tf(x) := \sin x f(x)$. (Here $f \in C^\infty(\mathbb{R})$ and $x \in \mathbb{R}$).

4. Example 2.14, problems (2.33) (which of course reads as $\varepsilon_n \to 0$) and (2.34). You may also consider the other examples (2.35), (2.36), though you may need to improve the proof of Th. 2.13.

5. Show that $\delta_0 \in \mathcal{D}'(\mathbb{R})$ is not a continuous positive function, i.e., $\delta_0 \neq I(f)$ for any continuous positive function f, for the embedding $I : L^1_{\text{loc}}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ constructed in the lecture notes. (Example b) on p.6). Can you actually show that δ_0 is not a locally integrable function?

6. Show that in $\mathcal{D}'(\mathbb{R}^2)$ we have

$$\Delta \log \frac{1}{r} = -2\pi\delta_0.$$

Here $r = |x|$, $x \in \mathbb{R}^2$.

7. Prove that in $\mathcal{D}'(\mathbb{R}^n)$, $n > 3$,

$$\Delta \frac{1}{r^{n-2}} = -(n-2)\lambda_n \delta_0,$$

where λ_n is the area of the unit ball of $\mathbb{R}^n$.

8. Prove that the sum

$$T = \sum_{j=1}^{\infty} \frac{\partial^j \delta_j}{\partial x^j}$$

converges in $\mathcal{D}'(\mathbb{R})$. Here δ_j is the Dirac measure of the point j. What is the order of the distribution T? Is it compactly supported?

9. Prove Theorem 2.20 of the lecture notes.

10. Show that if $f \in C^\infty(\mathbb{R})$ and $f(0) = 0$, then $f\delta_0 = 0$ in the space $\mathcal{D}'(\mathbb{R})$. In particular, $x\delta_0 = 0$.

11. Let $T \in \mathcal{D}'(\mathbb{R})$ and $f \in C^\infty(\mathbb{R})$. Is the following identity true (where f' is the classical derivative):

$$\frac{d(fT)}{dx} = f'T + f\frac{dT}{dx} \quad ?$$