Exercises, Fourier transform and Hausdorff dimension, spring 2014

1. Prove the Riemann-Lebesgue lemma: $\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0$ if $f \in L^1(\mathbb{R}^n)$.

In the following μ and ν are finite Borel measures on $\mathbb{R}^n$.

2. Let ψ be a non-negative continuous function on $\mathbb{R}^n$ with spt$\psi \subset B(0,1)$ and $\int \psi = 1$. Let $\psi_\epsilon(x) = \epsilon^{-n}\psi(x/\epsilon), x \in \mathbb{R}^n, \epsilon > 0$. Prove that $\psi_\epsilon * \mu \to \mu$ weakly and $\widehat{\psi_\epsilon * \mu} \to \widehat{\mu}$ as $\epsilon \to 0$.

3. Prove the formula $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$.

4. Prove the formula $\int \widehat{f\widehat{g}}d\mu = \int (\widehat{\mu} * \overline{g})f$.

5. What is the Fourier transform of the function f, as a distribution, when a) $f(x) = 1$ for all $x \in \mathbb{R}$, b) $f(x) = x$ for all $x \in \mathbb{R}$, c) $f(x) = x^2$ for all $x \in \mathbb{R}$?

6. What is the Fourier transform of the sign function $sgn; sgn(x) = 1$, when $x \geq 0$, $sgn; sgn(x) = -1$, when $x < 0$?

7. Show that if μ is a finite Borel measure on $\{x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $\hat{\mu}(\xi)$ does not tend to 0 as $|\xi| \to \infty$.

8. Show that there can be no inequality $\|\widehat{f\mu}\|_{L^q(\mathbb{R}^2)} \leq C_{p,q}\|f\|_{L^p(\mu)}$ for $1 \leq p \leq \infty, 1 \leq q < \infty$, where μ is the one-dimensional Lebesgue measure on the segment $[(-1,0),(1,0)]$.

9. What kind of decay estimate can you get for the Fourier transform of the surface measure on the conical surface

$$C = \{(z,t) \in \mathbb{R}^3 : |z| = t, 1 \leq t \leq 2\}$$

using the decay estimate for the length measure on the circle?

10. What kind of analog of the Tomas-Stein theorem can you get for the cone C?

11. Let μ be a finite Borel measure with compact support in $\mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^n$ an affine bijection ($Tx = Lx + a$ with L a linear bijection and $a \in \mathbb{R}^n$). Let

ν be the push-forward of μ under T; $\int g d\nu = \int g \circ T d\mu$. Show that if the restriction estimate

$$||\hat{f}||_{L^q(\mu)} \leq C_{p,q}||f||_p, \ f \in \mathcal{S}(\mathbb{R}^n),$$

holds for μ it holds also for ν. Conclude that the restriction theory for ellipsoids is essentially the same as for the sphere.

12. Let $R \subset \mathbb{R}^2$ be a rectangle with side-lengths r_1 and r_2 and ϕ a C^∞-function with support in R. Find a 'dual' rectangle $\tilde{R}$ of R with side-lengths $1/r_1$ and $1/r_2$ such that $\hat{\phi}$ decays very fast (what should this mean?) outside $\tilde{R}$.

13. Prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^\sigma(\mathbb{R}^n)$. The denseness of $C^\infty(\mathbb{R}^n)$ is proved in Lemma 17.1.

14. Prove that the operator norm of T_m as the operator from L^2 in L^2 equals $\|m\|_\infty$.

15. Prove that if $1 < p < \infty$ and $f \in L^p(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$f(x) = \lim_{R\to\infty} \int_{-R}^{R} e^{2\pi i x\xi} \widehat{f}(\xi) d\xi \text{ in the } L^p \text{ sense}$$

using the boudedness of the multiplier operators of the intervals $[-R, R]$.

16. Prove that if $1 < p < \infty$ and $n \geq 2$, then the convergence

$$f(x) = \lim_{R\to\infty} \int_{B(0,R)} e^{2\pi i x\cdot\xi} \widehat{f}(\xi) d\xi \text{ in the } L^p \text{ sense}$$

holds for all $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ if and only if $p = 2$.

17. Prove that for any $n \geq 1$, the operator norm of $T_{B(a,r)}$ is independent of $a \in \mathbb{R}^n$ and $r > 0$.

18. Prove that the characteristic function of a polyhedral domain $P \subset \mathbb{R}^n$ is an L^p multiplier for $1 < p < \infty$.

19. Prove that the Bochner-Riesz multiplier m_δ is not an L^p-multiplier if $p \leq \frac{2n}{n+1+2\delta}$ or $p \geq \frac{2n}{n-1-2\delta}$. You can of course use the kernel K_δ and its asymptotic properties.