Matematiikan ja tilastotieteen laitos
Exercises II for Fourier transfrom and Hausdorff dimension, spring 2014

Let $d(n)$ be the infimum of the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^n$.

1. Show that $d(n) \leq d(n+1)$.

2. Show that $d(m+n) \leq d(m)+n$.

3. Prove that there exists a Borel subset of $\mathbb{R}^2$ with positive Lebesgue measure whose orthogonal projections on all lines have empty interior. Hint: Consider Besicovitch sets.

4. Prove that there are circles in the plane centered at every point of a line segment covering only a set of measure zero. Hint: Let

$$B = \bigcup_{(a,b)\in C} \{(x,y) : (x-a)^2 + y^2 = a^2 + b\},$$

where C is as in the proof of Theorem 11.1 and modify that proof.

5. Fill the gap in the proof of Theorem 21.12, that is, $\|T_k f\|_{L^q(\mathbb{R}^n)}$ has the same estimate for $f \in \mathcal{S}(\mathbb{R}^n)$ without the assumption that $\mathrm{spt} f \subset B(0, 2^k)$.

6. Formulate and prove Theorem 22.9 for the Nikodym maximal function.

7. Prove that if μ is a finite Borel measure in $\mathbb{R}^n$ such that $\widehat{\mu} \in L^2$, then μ is absolutely continuos with Radon-Nikodym derivative in L^2.

8. Show that if μ and ν are finite Borel measures with compact support in $\mathbb{R}^n$, then $spt(\mu * \nu) \subset spt\mu + spt\nu$. Here $spt\mu = \{x : \mu(B(x,r)) > 0 \ \forall r > 0\}$ is the support of μ.

9. Prove that if G is a subgroup of $(\mathbb{R}, +)$ and there is a finite Borel measure μ with compact support such that $spt\mu \subset G$ and for some $\alpha > 0$, $|\widehat{\mu}(x)| \leq |x|^{-\alpha}$ for all $x \in \mathbb{R}$, then $G = \mathbb{R}$.