Differential Equations II
Exercise 4
3.4. 2014 (16-18 o'clock in CK111)

1. Compute the derivative $\overline{x}'(t)$, when the functions $\overline{x}(t)$ are defined by

$$(i)\ \overline{x}(t) = (e^{-2t}, te^{t})^T, \quad (ii)\ \overline{x}(t) = (1, e^t \cos t, e^t \sin t)^T.$$

2. Verify that $(\overline{x}^1(t), \overline{x}^2(t))$, where $\overline{x}^1(t) = e^{7t}(2,1)^T$ and $\overline{x}^2(t) = e^{-5t}(-2,1)^T$, forms a fundamental system of solutions of the homogeneous system

$$\overline{x}'(t) = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} \overline{x}(t).$$

3. Verify that $(\overline{x}^1(t), \overline{x}^2(t), \overline{x}^3(t))$ forms a fundamental system of solutions of

$$\overline{x}'(t) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix} \overline{x}(t),$$

where $\overline{x}^1(t) = e^t(1,0,-1)^T$, $\overline{x}^2(t) = e^{2t}(1,-1,-1)^T$ and $\overline{x}^3(t) = e^{-t}(1,2,-7)^T$.

4. Let $\overline{u} = (u_1, \ldots, u_n)^T \in \mathbf{R}^n$ be a given non-zero vector. Determine if the vector functions

$$(i)\ \overline{x}(t) = t\overline{u},\ t \in \mathbf{R}, \qquad (ii)\ \overline{x}(t) = \log(t)\overline{u},\ t \in (0,\infty),$$

can be a solution of a system $\overline{x}'(t) = A\overline{x}(t)$ of differential equations for some $n \times n$-matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ having constant coefficients.

5. Let $\alpha > 0$, $\beta > 0$ be constants. The system

$$x_1'(t) = -\alpha x_2(t)$$
$$x_2'(t) = -\beta x_1(t)$$

describes the amount of bacteria $x_1(t)$ and $x_2(t)$ at time $t \geq 0$ in a population where the species of bacteria consume each other without any regeneration. Compute $\frac{x_1(0)}{x_2(0)}$ when it is known that $x_1(t) > 0$ and $x_2(t) > 0$ for all $t \geq 0$.

6. Let $\overline{x}(t)$ be a solution of the homogeneous linear system

$$\overline{x}'(t) = A(t)\overline{x}(t)$$

in the open interval $I \subset \mathbf{R}$, where $A(t) = (a_{ij}(t)) \in \mathbf{R}^{n\times n}$. Show: if $\overline{x}(t_0) \neq \overline{0}$ for some $t_0 \in I$, then $\overline{x}(t) \neq \overline{0}$ for all $t \in I$. *Suggestion:* use the uniqueness theorem for linear systems.