Differential Equations II
Exercise 3
27.3. 2014 (16-18 o'clock room CK111)

1. Consider the Picard iterations in the proof of the local existence theorem,

$$y_0(x) = y_0, \quad y_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_k(t))dt, \quad k \in \mathbb{N},$$

applied to the initial value problem

$$y' = y + 2, \quad y(0) = 1.$$

Compute the functions y_1, y_2 and y_3.

2. Reduce the linear differential equation

$$y^{(4)} - 8y'' + 16y = \sin x$$

to a 1. order system in matrix form having 4 unknown functions.

3. Rewrite the system of differential equations

$$\begin{aligned} x'(t) &= -x(t) - 2y(t) + z(t) + e^t \\ y'(t) &= -2x(t) + y(t) - z(t) \\ z'(t) &= y(t) + 2z(t) - e^{-t} \end{aligned}$$

in matrix form.

4. The point (x_0, y_0) is an *equilibrium solution* of the system

$$\begin{aligned} x'(t) =& f_1(x(t), y(t)) \\ y'(t) =& f_2(x(t), y(t)) \end{aligned}$$

if the constant functions $x(t) = x_0$, $y(t) = y_0$ are solutions, that is, if $f_1(x_0, y_0) = 0 = f_2(x_0, y_0)$. Show that $(0, 0)$ is the only equilibrium solution of the linear system

$$\begin{aligned} x'(t) &= ax(t) + by(t) \\ y'(t) &= cx(t) + dy(t) \end{aligned}$$

if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$.

5. Verify that $x(t) = e^{-t} \cos t$, $y(t) = e^{-t} \sin t$, where $t \in \mathbf{R}$, solve the system

$$\begin{aligned} x'(t) &= -x(t) - y(t) \\ y'(t) &= x(t) - y(t). \end{aligned}$$

What happens to $(x(t), y(t))$ as $t \to \infty$?

6. Solve the linear system of differential equations

$$\begin{aligned} x'(t) &= x(t) + y(t) \\ y'(t) &= x(t) - y(t) \end{aligned}$$

by the elimination method (compare Example 5.2, pages 59-60, in the Differential Equations lecture notes). That is, differentiate the 1. equation, substitute y' from the second one and y from the first one to eliminate y.