Dependence logic

Problems 3

Tuesday 8.4.2014

1. Let $\phi(x_1,\ldots,x_n)$ and $\psi(x_1,\ldots,x_n)$ be formulas of dependence logic, with $x_1,\ldots,x_n$ appearing free, such that $\phi(x_1,\ldots,x_n) \equiv^* \psi(x_1,\ldots,x_n)$. For terms $t_1,\ldots,t_n$ let $\phi(t_1,\ldots,t_n)$ be the formula in which x_i is substituted by t_i for $1 \leq i \leq n$. The substitution is allowed only if no variable of t_i becomes bound in $\phi(t_1,\ldots,t_n)$. Show that for any terms $t_1,\ldots,t_n$ satisfying this condition for both ϕ and ψ,

$$\phi(t_1,\ldots,t_n) \equiv^* \psi(t_1,\ldots,t_n).$$

2. Show the following implication for all sentences ϕ, and models $\mathcal{M}$ and $\mathcal{M}'$: if $\mathcal{M} \cong \mathcal{M}'$, then $\mathcal{M} \models \phi \Leftrightarrow \mathcal{M}' \models \phi$. Prove the claim using induction on ϕ for all formulas. Consider the cases where $\phi = \psi_1 \vee \psi_2$ or $\phi = \exists x_n \psi$.

3. Find a logically equivalent first-order formula in each case (below f and P are symbols of the vocabulary):

1. $\exists x_0 (=\!(x_1, x_0) \wedge (f(x_1) = x_1))$
2. $\exists x_0 (=\!(x_2, x_0) \wedge (P(f(x_0)) \wedge \neg P(x_1)))$

4. Which of the following formulas are logically equivalent to a first-order formula:

1. $=\!(x_0, x_1, x_2) \wedge x_0 = x_1$
2. $(=\!(x_0, x_2) \wedge x_0 = x_1) \rightarrow =\!(x_1, x_2)$
3. $\forall x_0 \exists x_2 (=\!(x_0, x_2) \wedge x_2 = x_1)$

5. Let $L = \emptyset$ and $M = \{0, 1\}$. Can you find a first-order formula ψ such that for all teams X of M with domain $\{x_0, x_1, x_2\}$: $M \models_X \psi \Leftrightarrow M \models_X \phi$, when

- $\phi := \exists x_0 (=\!(x_2, x_0) \wedge \neg(x_0 = x_1))$, or
- $\phi := \exists x_0 (=\!(x_2, x_0) \wedge (x_0 = x_1 \vee x_0 = x_2))$.

6. Let ϕ be the formula $\exists x_0 \forall x_1 \neg(=\!(x_2, x_1) \wedge (x_0 = x_1))$. Show that the flattening of ϕ is not a strong logical consequence of ϕ.