Dependence logic

Problems 1

Tuesday 25.3.2014

1. For a first-order formula ϕ, consider the operations $\phi \mapsto \phi^p$ and $\phi \mapsto \phi^d$ that transform ϕ to a formula in negation normal form. Show that

$$(\phi^d)^d = (\phi^p)^p = \phi^p,$$

and that ϕ^p is logically equivalent to ϕ and ϕ^d to $\neg\phi$.

2. Let $M = \{0, 1, 2\}$. Consider the following team X of M with domain $\{x_0, x_1, x_2\}$:

	x_0	x_1	x_2
s_0	1	2	2
s_1	2	1	2
s_2	0	1	2

Is X of type ϕ (that is, does $M \models_X \phi$ hold) if:

1. $\phi := x_0 = x_2$ or $\phi := \neg x_0 = x_2$
2. $\phi := \exists x_0(x_0 = x_2)$
3. $\phi := \forall x_3 {=}(x_2)$
4. $\phi := ({=}(x_0, x_1) \vee {=}(x_1, x_2))$

3. Let $\mathcal{M} = (\mathbb{N}, +, \times, 0, 1)$, where $\mathbb{N} = \{0, 1, \ldots\}$. Which teams X of $\mathcal{M}$ with domain $\{x_0, x_1\}$ are of type

1. ${=}(x_0, x_0 + x_1)$
2. ${=}(x_0 \times x_0, x_1 \times x_1)$

4. Let ϕ be the formula $({=}(x_0, x_1) \vee \neg x_0 = x_1)$. Show that for all $\mathcal{M}$ and $X \neq \emptyset$:

$$\mathcal{M} \models_X \neg\phi \Leftrightarrow \mathcal{M} \not\models_X \phi.$$

5. Show that $\models \forall x_0 \forall x_1 (x_1 = c \rightarrow {=}(x_0, x_1))$.

6. The Closure Test shows that if $Y \subseteq X$ and $\mathcal{M} \models_X \phi$ then $\mathcal{M} \models_Y \phi$, for all formulas ϕ, models $\mathcal{M}$, and teams X. Complete the inductive proof of this result by considering the cases where ϕ is of the form $\neg\psi$ or $\exists x_n \psi$.