P55

① Let $\phi \in D$ be a sentence defining infinity i.e. for all $\mathcal{M}$, $\mathcal{M} \models \phi \iff |M|$ is infinite.

Consider now $\neg\phi$. We claim that for no ψ of the form

$$\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m (\theta_1 \wedge \theta_2) \qquad (1)$$

where θ_1 is a conjunction of dep.atoms and $\theta_2 \in FO$, it holds that $\neg\phi \equiv^* \psi$. Assume to the contrary that such ψ exists. Then $\mathcal{M} \models_{\{\emptyset\}} \neg\psi \iff \mathcal{M} \models_{\{\emptyset\}} \exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m$ $(\neg\theta_1 \vee \neg\theta_2) \iff \mathcal{M} \models_{X'} \neg\theta_1 \vee \neg\theta_2$ where $X' := \{\emptyset\}[F/\bar{x}][M^m/\bar{y}]$ for some $F: \{\emptyset\} \to M^n$ $\overset{(*)}{\iff} \mathcal{M} \models_{X'} \neg\theta_2$ where $X' \ldots \iff \mathcal{M} \models_{\{\emptyset\}} \exists x_1 \ldots \exists x_n$ $\forall y_1 \ldots \forall y_m \neg\theta_2$.

(For $(*)$ note that $\theta_1 = \bigwedge_{i=1}^{l} d_i$ where each d_i is a dep.atom. Hence $\neg\theta_1 \equiv \bigvee_{i=1}^{l} \neg d_i$ is of type X iff $X = \emptyset$.)

Now by the counter assumption $\neg\neg\phi \equiv \neg\psi$ when

$$\phi \equiv \neg\neg\phi \equiv \neg\psi \equiv \exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m \neg\theta_2 \in FO.$$

Since infinity is not definable in FO, this is a contradiction. Hence the counter assumption is false and no such ψ exists. □

PS5

in NNF and with no $\forall$

(2) Let ϕ be a sentence of D and let $\phi^* \in FO$ be obtained from ϕ by replacing each dependence atom with $\top$. We show by structural induction that for all assignments s, and for all formulas ϕ of the above form,

$$\mathcal{M} \models_{\{s\}} \phi \iff \mathcal{M} \models_{\{s\}} \phi^*.$$

(i) ϕ atomic or negated atomic: If $\phi = \alpha$ or $\phi = \neg\alpha$ where $\alpha \in FO$ is atomic, then $\phi^* = \phi$ and the claim clearly holds. If $\phi = d$ where d is a dep. atom, then $\phi^* = \top$ and for all s, $\mathcal{M} \models_{\{s\}} d$ and $\mathcal{M} \models_{\{s\}} \top$. If $\phi = \neg d$, then $\phi^* = \neg\top$ and for all s, $\mathcal{M} \not\models_{\{s\}} \neg d$ and $\mathcal{M} \not\models_{\{s\}} \neg\top$.

(ii) $\phi = \psi_0 \vee \psi_1$: Then for all s, $\mathcal{M} \models_{\{s\}} \phi \iff \mathcal{M} \models_{\{s\}} \psi_0$ or $\mathcal{M} \models_{\{s\}} \psi_1 \overset{\text{ia}}{\iff} \mathcal{M} \models_{\{s\}} \psi_0^*$ or $\mathcal{M} \models_{\{s\}} \psi_1^* \iff \mathcal{M} \models_{\{s\}} \phi^*$.

(iii) $\phi = \psi_0 \wedge \psi_1$: As case (ii).

(iv) $\phi = \exists x \psi$: Then for all s, $\mathcal{M} \models_{\{s\}} \phi \iff$ for some $a \in M$ $\mathcal{M} \models_{\{s(a/x)\}} \psi \overset{\text{ia}}{\iff} \mathcal{M} \models_{\{s(a/x)\}} \psi^* \iff \mathcal{M} \models_{\{s\}} \phi^*$.

This concludes the induction proof. Hence for all $\mathcal{M}$,

$$\mathcal{M} \models \phi :\iff \mathcal{M} \models_{\{\emptyset\}} \phi \iff \mathcal{M} \models_{\{\emptyset\}} \phi^* \iff: \mathcal{M} \models \phi^*$$

[Also then by Locality (L.3.27), for all $\mathcal{M}$ and X, $\mathcal{M} \models_X \phi \iff \mathcal{M} \models_X \phi^*$.] $\square$

(3) $=(x_1,\dots,x_n) \equiv x_n \perp_{x_1\dots x_{n-1}} x_n$:

"$\Rightarrow$" Assume $M \models_X =(x_1,\dots,x_n)$ and let $s,s' \in X$ be s.t. $s(x_1\dots x_{n-1}) = s'(x_1\dots x_{n-1})$. By the as. $s(x_n) = s'(x_n)$, when we can choose $s'' := s$.

"$\Leftarrow$" Assume $M \models_X x_n \perp_{x_1\dots x_{n-1}} x_n$ and let $s,s' \in X$ be s.t. $s(x_1\dots x_{n-1}) = s'(x_1\dots x_{n-1})$. Then by the as. there is $s'' \in X$ s.t. $s''(x_1\dots x_n) = s(x_1\dots x_n)$ and $s''(x_1\dots x_n) = s'(x_1\dots x_n)$. Then $s(x_n) = s'(x_n)$. $\square$

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(4) Let S be an n-ary relation symbol encoding team X where $Dom(X) = \{x_1,\dots,x_n\} = Fr(\phi)$.

- If $\phi = x_{i_1}\dots x_{i_l} \subseteq x_{j_1}\dots x_{j_l}$, then we define

$$T_{1,\phi} := \forall x_1\dots x_n \exists y_1\dots y_n \left[S(x_1\dots x_n) \rightarrow \left(S(y_1\dots y_n) \wedge \bigwedge_{k=1}^{l} x_{i_k} = y_{i_k} \right) \right].$$

- If $\phi = \bar{b} \perp_{\bar{a}} \bar{c}$, then we define

$$T_{1,\phi} := \forall x_1\dots x_n \forall y_1\dots y_n \exists z_1\dots z_n \left[\left(S(\bar{x}) \wedge S(\bar{y}) \wedge \bar{a} = \bar{a}_y \right) \rightarrow \left(S(\bar{z}) \wedge \bar{a}_z = \bar{a} \wedge \bar{b}_z = \bar{b} \wedge \bar{c}_z = \bar{c}_y \right) \right]$$

where $\bar{a}_y$ is obtained from $\bar{a}$ by substituting $x_i \mapsto y_i$, for $i=1,\dots,n$. ($\bar{a}_z, \bar{b}_z, \bar{c}_z$ defined analogously by substituting $x_i \mapsto z_i$). $\square$

(5) We claim

$$\underbrace{\bigwedge_{i=1}^{k-1} =(t_i) \twoheadrightarrow =(t_k)}_{\phi} \quad \equiv \quad =(t_1, \dots, t_k).$$

"$\Rightarrow$" Assume $M \models_X \phi$ and let $s, s' \in X$ be s.t.

$t_1^M\langle s\rangle = t_1^M\langle s'\rangle, \dots, t_{k-1}^M\langle s\rangle = t_{k-1}^M\langle s'\rangle$.

Then $M \models_{\{s,s'\}} \bigwedge_{i=1}^{k-1} =(t_i)$ when by the assumption

$M \models_{\{s,s'\}} =(t_k)$. Hence $t_k^M\langle s\rangle = t_k^M\langle s'\rangle$ when

we obtain that $M \models_X =(t_1, \dots, t_k)$.

"$\Leftarrow$" Assume $M \models_X =(t_1, \dots, t_k)$, and let $Y \subseteq X$ be s.t.

$M \models_Y \bigwedge_{i=1}^{k-1} =(t_i)$. If $Y = \emptyset$, then $M \models_Y =(t_k)$.

If not, then take $s \in Y$ and define $a := t_k^M\langle s\rangle$.

Let $s' \in Y$ be arbitrary. Since (by Closure Test)

$M \models_{\{s,s'\}} \bigwedge_{i=1}^{k-1} =(t_i)$, we obtain that $t_1^M\langle s\rangle = t_1^M\langle s'\rangle, \dots,$

$t_{k-1}^M\langle s\rangle = t_{k-1}^M\langle s'\rangle$. Then by the assumption,

$t_{k-1}^M\langle s'\rangle = t_{k-1}^M\langle s\rangle = a$. Hence for all $s' \in Y$,

$t_{k-1}^M\langle s'\rangle = a$ when we conclude that $M \models_Y =(t_k)$. $\square$

⑥ Define $\psi := \phi \twoheadrightarrow \bot$.

Now $M \models_{\{\emptyset\}} \psi \iff$ for all $Y \subseteq \{\emptyset\}$: if $M \models_Y \phi$, then $M \models_Y \bot \iff M \not\models_{\{\emptyset\}} \phi$.

For this, note the empty set property is preserved for $D(\twoheadrightarrow)$ also, that is, for all $\theta \in D(\twoheadrightarrow)$, $M \models_\emptyset \theta$. $\square$