PS4

(1) For example,

$$\phi := \exists c \forall x \exists y \forall a \exists b \Big[\overset{(1)}{x = c} \lor \Big(\overset{(2)}{x \neq c} \land \overset{(3)}{=(a,b)} \land \overbrace{x = a \rightarrow y = b}^{(4)}$$
$$\land \overbrace{y = a \rightarrow x = b}^{(5)} \land \overset{(6)}{x \neq y} \land \overset{(7)}{y \neq c} \Big) \Big]$$

ϕ describes that we can choose one element $c \in M$ such that in $M \setminus \{c\}$ there is a pairing function i.e. a function $f : M \setminus \{c\} \rightarrow M \setminus \{c\}$ s.t. $\forall x \in M \setminus \{c\} : f(x) \neq x$ and $f(f(x)) = x$. Now $x \mapsto y$ is this function. (3) says that $a \mapsto b$ is also a function. (4) says that $x \mapsto y$ and $a \mapsto b$ are the same functions. (5) ensures that $f(f(x)) = x$ and (6) that $f(x) \neq x$ and (7) that $f(x) \in M \setminus \{c\}$.

□

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(2) Let

$$\psi := \exists a \exists b \forall x \exists y \forall x' \exists y' \Big(\overset{(1)}{a \neq b} \land \overbrace{(y = a \lor y = b)}^{(2)} \land \overset{(3)}{=(x', y')}$$
$$\land \overbrace{x = x' \rightarrow y = y'}^{(4)} \land \overbrace{Exx' \rightarrow y \neq y'}^{(5)} \Big)$$

For G s.t. $\underbrace{|V|}_{\text{set of vertices}} \geq 2$, $G \models \psi \Leftrightarrow G$ is 2-colorable.

Here (1) says that a and b are two distinct elements of V, intended to encode subsets $A, B \subseteq V$ s.t. $A \cap B = \emptyset$, $A \cup B = V$, $\forall x x' \in V : (Ax \land Ax' \rightarrow \neg Exx') \land (Bx \land Bx' \rightarrow \neg Exx')$. (2) now says that $A \cup B = V$, (1) ensures that $A \cap B = \emptyset$, (3) that $x' \mapsto y'$ is a function and (4) that is the same function that $x \mapsto y$. (5) describes that if Exx', then $\{x, x'\} \not\subseteq A$ and $\{x, x'\} \not\subseteq B$.

Since ψ works only for graphs with at least 2 vertices, we define

$\phi := \forall x \forall y\, x = y \lor (\exists x \exists y\, x \neq y \land \psi)$ when for all G,

$$G \models \phi \Leftrightarrow G \text{ is 2-colorable.}$$

□

③ We show that if $\models \phi \to \psi$, then $\begin{cases} \phi \Rightarrow \psi, & (i) \\ \neg\psi \Rightarrow \neg\phi. & (ii) \end{cases}$

Assume that $\models \phi \to \psi$ (1). Then

(i) Assume that $\mathcal{M} \models_X \phi$. By (1) $\mathcal{M} \models_X \neg\phi \vee \psi$. Let $Y \cup Z = X$ be s.t. $\mathcal{M} \models_Y \neg\phi$ and $\mathcal{M} \models_Z \psi$. By Closure test $\mathcal{M} \models_Y \phi$. Since $\theta \Rightarrow \theta^f$ (P 3.40), $\mathcal{M} \models_Y (\neg\phi)^f$ and $\mathcal{M} \models_Y \phi^f$. Also $(\neg\phi)^f = \neg\phi^f$. If now $Y \neq \emptyset$, by C 3.32 we would find $s \in Y$ s.t. $\mathcal{M} \models_s \phi^f$ and $\mathcal{M} \models_s \neg\phi^f$. This cannot be the case; therefore $Y = \emptyset$ when $Z = X$ when $\mathcal{M} \models_X \psi$.

(ii) Assume that $\mathcal{M} \models_X \neg\psi$. By (1) there are $Y \cup Z = X$ s.t. $\mathcal{M} \models_Y \neg\phi$ and $\mathcal{M} \models_Z \psi$. Analogously to the previous case, $Z = \emptyset$ when $\mathcal{M} \models_X \neg\phi$. □

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④ Let $\alpha := P(x_0)$ when $\phi = \exists x_0 \alpha$.

Now $T_{1,\alpha}(R) = \forall x_0 (R(x_0) \to P(x_0))$,

and $T_{1,\phi} = \exists R (T_{1,\alpha}(R) \wedge \top \to \exists x_0 R(x_0))$. □

Since in $T_{1,\phi}$, S is 0-ary we replace S with $\top$.

(Note that now $\mathcal{M} \models_{\{\emptyset\}} T_{1,\phi} \Leftrightarrow \mathcal{M} \models \phi$

but $\mathcal{M} \models_{\emptyset} T_{1,\phi} \not\Leftrightarrow \mathcal{M} \models \phi$ since $\mathcal{M} \models_{\emptyset} T_{1,\phi}$

always.)

(5) We assume that x_n appears free in φ. Otherwise $\forall x_n \phi \equiv \phi \in \Sigma_1^1$.

Then $\forall x_n \phi \equiv \exists P \forall x_n \varphi'$ where P is $k+1$-ary and φ' is obtained from φ by replacing each $R t_1 \ldots t_k$ with $P t_1 \ldots t_k x_n$.

Proof:

"$\forall x_n \phi \Rightarrow \exists P \forall x_n \varphi'$"

Assume that $M \models \forall x_n \phi$. Then for all $a \in M$ there is $R_a^M \subseteq M^k$ s.t. $(M, R_a) \models_{(a/x_n)} \varphi$. We let $P^M := \bigcup_{a \in M} R_a \times \{a\}$.

when for all $a \in M$, $(M, P^M) \models_{(a/x_n)} \varphi'$. For this, it suffices to prove by strutural induction that for all $a \in M$, for all s and for all $\theta \in FO^{(*)}$,

$$(M, R_a) \models_{s(a/x_n)} \theta \iff (M, P^M) \models_{s(a/x_n)} \theta'$$

where θ' is obtained from θ as φ' is from φ.

(other cases are straightforward)

In the atomic case where $\theta = R t_1 \ldots t_k$, we have

$$\left(t_1^M\langle s(a/x_n)\rangle, \ldots, t_k^M\langle s(a/x_n)\rangle\right) \in R_a \iff \left(t_1^M\langle s(a/x_n)\rangle, \ldots, t_k^M\langle s(a/x_n)\rangle, a\right) \in P^M.$$

By the induction proof we obtain that $(M, P^M) \models_{(a/x_n)} \varphi'$ for all $a \in M$. Hence $M \models \exists P \forall x_n \varphi'$.

(*) Here we assume that θ is such that x_n does not occur bound in θ, (since wlog this holds for φ).

⑤ "$\exists P \forall x_n \psi' \Rightarrow \forall x_n \phi$"

cont.

Assume that $M \models \exists P \forall x_n \psi'$. Then there is $P^M \subseteq M^{k+1}$ s.t. for all $a \in M$, $(M, P^M) \models_{(a/x_n)} \psi'$. Now for all $a \in M$, we choose $R_a := \{(b_1, \dots, b_k) \in M^k : (b_1, \dots, b_k, a) \in P^M\}$.

We need to show that for all $a \in M$, $(M, R_a) \models_{(a/x_n)} \psi$.

This follows from the induction proof of the previous page. $\square$

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⑥ Let $\phi \in D$ and and assume that ϕ is true in arbitrarily large finite models or in a model of infinite cardinality. (1) Let $\exists P_1 \dots \exists P_k \psi \in \Sigma_1^1$ be equivalent to ϕ.

Then (1) is true for $\exists P_1 \dots \exists P_k \psi$ and also for ψ.

Then by FO L-S, for all cardinalities κ, there exists $(M, P_1^M, \dots, P_k^M) \models \psi$ s.t. $|M| = \kappa$. when $M \models \exists P_1 \dots \exists P_k \psi$ and $M \models \phi$. $\square$