[PS 3]

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As. $\phi(x_1, \ldots, x_n) \equiv^T \psi(x_1, \ldots, x_n)$.

Let $\mathcal{M}, X$ be s.t. $\mathrm{Dom}(X) \supseteq \mathrm{Var}(\bar{t})$.

① $\mathcal{M} \models_X \phi(t_1, \ldots, t_n) \overset{(*)}{\iff} \mathcal{M} \models_{X'} \phi(x_1, \ldots, x_n)$

(where $X' := \{ s : \{x_1, \ldots, x_n\} \to \mathcal{M} \mid s(x_i) = t_i^{\mathcal{M}}\langle u \rangle \ \forall i = 1, \ldots, n$ for some $u \in X \}$.)

$\overset{\text{as.}}{\iff} \mathcal{M} \models_{X'} \psi(x_1, \ldots, x_n) \overset{(*)}{\iff} \mathcal{M} \models_X \psi(t_1, \ldots, t_n)$.

Also, $\mathcal{M} \models_X \neg\phi(t_1, \ldots, t_n) \overset{(*)}{\iff} \mathcal{M} \models_{X'} \neg\phi(x_1, \ldots, x_n)$

$\overset{\text{as.}}{\iff} \mathcal{M} \models_{X'} \neg\psi(x_1, \ldots, x_n) \overset{(*)}{\iff} \mathcal{M} \models_X \neg\psi(t_1, \ldots, t_n)$.

□

For (*) see next 4 pages where this is showed by induction on complexity of ϕ. Note that here we apply the case where $\bar{y}$ is the empty sequence.

Claim:

Let $\bar{x} := (x_1, \dots, x_n)$ and $\bar{t} := (t_1, \dots, t_n)$. Let $\phi(\bar{x}) \in D$ with free variables $\bar{x}$, and let $\phi(\bar{t})$ be obtained from ϕ by substituting $x_i \to t_i$ for every free occurrence of x_i. Assume that no variable in t_i become bound in the substitution. Let $\mathcal{M}$ be a model, and X a team such that $\mathrm{Dom}(X) = \mathrm{Var}(\bar{t})$. Then

(i) $\mathcal{M} \models_X \phi(\bar{t}) \iff \mathcal{M} \models_{X'} \phi(\bar{x})$,

(ii) $\mathcal{M} \models_X \neg\phi(\bar{t}) \iff \mathcal{M} \models_{X'} \neg\phi(\bar{x})$,

where, for a team Y of $\mathcal{M}$ with $\mathrm{Dom}(Y) = \mathrm{Var}(\bar{t})$, we let

$$Y' := \{ s : \{x_1, \dots, x_n\} \to M \mid \exists u \in X \, [s(\bar{x}) = \bar{t}^{\mathcal{M}}(u)] \}$$

Proof of Claim: By structural induction.

atom: Easy.

negation: Easy by the induction assumption.

disjunction: Assume that $\phi(\bar{x}) = \alpha(\bar{x}) \vee \beta(\bar{x})$

(i) $\mathcal{M} \models_X \phi(\bar{E}) \iff \mathcal{M} \models_Y \alpha(\bar{E})$ and $\mathcal{M} \models_Z \beta(\bar{E})$ for some $Y \cup Z = X \overset{ia}{\underset{*}{\iff}} \mathcal{M} \models_{Y'} \alpha(\bar{x})$ and $\mathcal{M} \models_{Z'} \beta(\bar{x})$ for some $Y \cup Z = X \overset{**}{\iff} \mathcal{M} \models_A \alpha(\bar{x})$ and $\mathcal{M} \models_B \beta(\bar{x})$ for some $A \cup B = X' \iff \mathcal{M} \models_{X'} \alpha(\bar{x}) \vee \beta(\bar{x})$.

(ii) Analogous but easier

existential: Assume that $\phi(\bar{x}) = \exists y \, \psi(\bar{x}, y)$. Since $\bar{x}$ lists free variables of ϕ, $y \notin \bar{x}$.

(i) Assume first that $\mathcal{M} \models_X \phi(\bar{E})$. Then $\exists F : X \to M$ s.t. $\mathcal{M} \models_{X[F/y]} \psi(\bar{E}, y)$ By the ind. as. (note *), applied to the substitution $\bar{x}y \mapsto \bar{E}y$,

$$\mathcal{M} \models_{X[F/y]'} \psi(\bar{x}, y). \qquad (1)$$

* Without loss of generality we may assume that α and β have exactly variables listed in $\bar{x}$ as their free variables,

** For "$\Rightarrow$" we choose $A := Y'$ and $B := Z'$ when $Y \cup Z = X'$. For "$\Leftarrow$" we choose e.g. $Y := \{u \in X \mid \exists s \in A \, (s(\bar{x}) = \bar{E}^{\mathcal{M}}\langle u \rangle)\}$.

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We now let $G : X' \to M$ be s.t. $G(s) = F(u)$ where $u \in X$ is some choice such that $s(\bar{x}) = \bar{t}^{M}\langle u \rangle$.

Then $X'[G/y] \subseteq X[F/y]'$. For this, note that y does not appear in $\bar{t}$ (*), and hence not in $\mathrm{Dom}(X)$.

By the above and (1), we conclude by Closure Test that $\mathcal{M} \models_{X'[G/y]} \psi(\bar{x}, y)$ when $\mathcal{M} \models_{X'} \phi(\bar{x})$.

"$\Leftarrow$" Assume that $\mathcal{M} \models_{X'} \phi(\bar{x})$ when $\mathcal{M} \models_{X'[G/y]} \psi(\bar{x}, y)$ for some $G : X' \to M$.

Now, for each $u \in X$, there is exactly one $s_u \in X'$ s.t. $s_u(\bar{x}) = \bar{t}^{M}\langle u \rangle$. We let $F(u) := G(s_u)$, for $u \in X$, when $X'[G/y] = X[F/y]'$.

By the idd. as. then $\mathcal{M} \models_{X[F/y]} \psi(\bar{t}, y)$ when $\mathcal{M} \models_{X} \phi(\bar{t})$.

(ii) Analogous but easier.

(*) Since we assume that no variable in $\bar{t}$ become bound in the subst.

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Hence we have showed (Claim). Note that we had the restriction that $\mathrm{Dom}(X) = \mathrm{Var}(F)$. However, by Locality (Lemma 3.27) we now obtain the result of the 1st page. □

(can be proved independently)

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Claim:

(2) For all $M \cong M'$, all $\pi: M \to M'$, all X of M and all $\phi \in D$,

(i) $M \models_X \phi \iff M' \models_{\pi X} \phi$, where $\pi X := \{\pi \circ s : s \in X\}$.

(ii) $M \models_X \neg\phi \iff M' \models_{\pi X} \neg\phi$,

Proof: Induction on complexity of ϕ.

$\boxed{\vee}$ See page 36 of Jouko's book.

$\boxed{\exists}$ Assume that $\phi = \exists y \psi$.

(i) "$\Rightarrow$" $M \models_X \phi \Rightarrow M \models_{X[F/y]} \psi$ for a $F: X \to M$

$\overset{\text{IH}}{\Rightarrow} M' \models_{\pi(X[F/y])} \psi$ for a $F: X \to M$

$\Rightarrow M' \models_{(\pi X)[G/y]} \psi$ where $G(s) = \pi(F(\pi^{-1} \circ s))$ for a $F: X \to M$.

(since $\pi(X[F/y]) = (\pi X)[G/y]$.)

$\Rightarrow M' \models_{\pi X} \phi$.

"$\Leftarrow$" We choose $\overset{M' \cong M,}{\pi^{-1}}$ and πX and apply the above:

$M' \models_{\pi X} \phi \Rightarrow M \models_{\pi^{-1}\pi X} \phi \Rightarrow M \models_X \phi$.

② cont...

(ii) "$\Rightarrow$" $M \nvDash_{x} \neg\phi \Rightarrow M \vDash_{x[M/y]} \neg\psi$

$\overset{i.}{\Rightarrow} M' \nvDash_{\pi(x[M/y])} \neg\psi \Rightarrow M' \vDash_{(\pi x)[M/y]} \neg\psi$

(since $\pi(x[M/y]) = (\pi x)[M/y]$.)

$\Rightarrow M' \nvDash_{\pi x} \neg\phi$.

"$\Leftarrow$" We choose $\overset{M' \equiv M}{\pi^{-1}}$ and πx and apply the above:

$M' \nvDash_{\pi x} \neg\phi \Rightarrow M \vDash_{\pi^{-1}\pi x} \neg\phi \Rightarrow M \nvDash_{x} \neg\phi$.

$\square$

(3) RÉGION AQUITAINE

1. Let M, X be arbitrary. Then, if $\phi := f(x_1) = x_1$,

$M \models_X \exists x_0 \overbrace{(=(x_1, x_0) \wedge (f(x_1) = x_1))}^{\psi} \iff M \models_{X[F/x_0]} \psi$

for a $F: X \to M$ $\Rightarrow \exists F \forall s \in X[F/x_0]: f(s(x_1)) = s(x_1)$

$\iff \forall s \in X: f(s(x_1)) = s(x_1) \iff M \models_X f(x_1) = x_1$.

For the other direction, assume ~~that~~ $M \models_X \phi$, and let $F: X \to M$ be such that $F(s) = 0$ for all $s \in X$ where $0 \in M$. Then clearly $M \models_{X[F/x_0]} =(x_1, x_0)$. Also, for all $s \in X[F/x_0]$, $f(s(x_1)) = s(x_1)$ by the assumption. Hence $M \models_{X[F/x_0]} \psi$ when $M \models_X \exists x_0 \psi$.

2. Let M, X be arbitrary, and let $\phi := \exists x_0 (P(f(x_0)) \wedge \neg P(x_1))$.

Clearly $\exists x_0 \overbrace{(=(x_2, x_0) \wedge (P(f(x_0)) \wedge \neg P(x_1)))}^{\psi} \Rightarrow \phi$.

For the other direction, assume that $M \models_X \phi$ (1). Let $F: X \to M$ be a function witnessing this i.e. $M \models_{X[F/x_0]} P(f(x_0)) \wedge \neg P(x_1))$. Let $a = F(s)$ for some $s \in X$. By (1) $f(a) \in P^M$ and for all $s \in X$: $s(x_1) \notin P^M$. Hence we obtain that $M \models_{X[\{a\}/x_0]} P(f(x_0)) \wedge \neg P(x_1))$. Clearly $M \models_{X[\{a\}/x_0]} =(x_2, x_0)$.

Hence the constant function $G(s) = a$ $\forall s \in X$ is a witness of $M \models_X \exists x_0 \psi$. □

Let $M = (\{0,1\})$. Let $X :=$

	x_0	x_1	x_2
s	0	0	0
s'	0	0	1

(4) 1. No. Let $\phi_1 := =(x_0, x_1, x_2) \wedge x_0 = x_1$ and assume that ψ_1 is a FO formula such that $\phi_1 \equiv \psi_1$. Then, by the assumption, $M \not\models_X \psi_1$ but $M \models_{\{s\}} \psi_1$ and $M \models_{\{s'\}} \psi_1$, when $M \models_s \psi_1$ for all $s \in X$. Hence ψ_1 is not flat which is a contradiction. ↯ Therefore, no such ψ_1 exists.

2. No. Let $\phi_2 := (=(x_0, x_2) \wedge x_0 = x_1) \rightarrow =(x_1, x_2)$ and assume that ψ_2 is a FO formula s.t. $\phi_2 \equiv \psi_2$.

Now, $M \not\models_X \phi_2$: For this, assume to the contrary that $Y \cup Z = X$ are such that $M \models_Y \neg(=(x_0, x_2) \wedge x_0 = x_1)$ and $M \models_Z =(x_1, x_2)$. Then there are $Y_1 \cup Y_2 = Y$ such that $M \models_{Y_1} \neg =(x_0, x_2)$ and $M \models_{Y_2} \neg x_0 = x_1$. But now from this it follows that $Y_1 = Y_2 = \emptyset$. Hence we have that $Z = X$ when $M \models_X =(x_1, x_2)$. This is a contradiction. ↯ Hence $M \not\models_X \phi_2$ when by the assum $M \not\models_X \psi_2$. Then again, it is straightforward to check that $M \models_{\{s\}} \phi_2$ and $M \models_{\{s'\}} \phi_2$ when by the assumption $M \models \psi_2$ for all $s \in X$. ↯ with flatness of ψ_2. Hence no such ψ_2 exists.

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(4) cont...

Let $X' :=$

	x_1
s	0
s'	1

.

3. No. Let $\phi_3 := \forall x_0 \exists x_2 \big(\overbrace{=\!(x_0, x_2) \wedge x_2 = x_1}^{\theta}\big)$, and assume that ψ_3 is a FO formula s.t. $\phi_3 \equiv \psi_3$.

Now, $M \not\models_{X'} \phi_3$. For this, assume to the contrary that there exists $F : X[M/x_0] \to M$ such that

$M \models_{X''} \theta$ (1) where $X'' := X[M/x_0][F/x_2]$.

Now $s(0/x_0), s'(0/x_0) \in X[M/x_0]$ and $\overbrace{s(0/x_0)(a/x_2)}^{t}$, $\overbrace{s'(0/x_0)(b/x_2)}^{t'} \in X''$ where $a := F(s(0/x_0))$ and $b := F(s'(0/x_0))$.

By (1) $M \models_t x_2 = x_1$ and $M \models_{t'} x_2 = x_1$ when $a = s(x_1) = 0$ and $b = s'(x_1) = 1$. Then $t(x_0) = t'(x_0)$ but $t(x_2) \neq t'(x_2)$.

Hence $M \not\models_{X''} =\!(x_0, x_2)$ which is ↯. Hence we obtain that $M \not\models_{X'} \phi_3$. Again, since $M \models_{\{s\}[M/x_0][\{0\}/x_2]} \theta$

and $M \models_{\{s'\}[M/x_0][\{1\}/x_2]} \theta$, we obtain that

$M \models_{\{s\}} \phi_3$ and $M \models_{\{s'\}} \phi_3$ when by the as. $M \models \psi_3$

for all $s \in X$, ↯ by flatness of ψ_3 being first-order.

Hence no such ψ_3 exists. □

(5) 1. No. Let $X :=$

	x_0	x_1	x_2
s	0	1	0
s'	1	0	0

. Then $M \not\models_X \phi$.

Otherwise, there would be $F: X \to M$ such that $M = \{0,1\}$ and $M \models_{X[F/x_0]} =(x_2, x_0) \wedge \neg(x_0 = x_1)$. Then, by $M \models_{X[F/x_0]} \neg(x_0 = x_1)$, $F(s) = 0$ and $F(s') = 1$ when $X[F/x_0] = X$. But $M \not\models_X =(x_2, x_0)$. Hence we obtain that $M \not\models_X \phi$.

Clearly, $M \models_{\{s\}} \phi$ and $M \models_{\{s'\}} \phi$. Now, if $\varphi \in FO$ would satisfy the claim, then $M \not\models_X \varphi$ and $M \models_s \varphi$ $\forall s \in X$ which would contradict with flatness of φ. Hence no such φ exists.

2. Yes. We let $\varphi := \top$. For the claim it suffices to show that for all X of M with domain $\{x_0, x_1, x_2\}$, $M \models_X \exists x_0 (=(x_2, x_0) \wedge (x_0 = x_1 \vee x_0 = x_2))$. So let X be arbitrary of this kind. We define $F(s) = s(x_2)$ for all $s \in X$. Then $M \models_{X[F/x_2]} x_0 = x_2$ when $M \models_{X[F/x_2]} x_0 = x_1 \vee x_0 = x_2$. Also, $M \models_{X[F/x_2]} =(x_2, x_0)$ clearly. Hence we have showed the claim. $\square$

⑥ $\phi_f = \exists x_0 \forall x_1 \neg \overbrace{(\top \wedge (x_0 = x_1))}^{\psi}$. Let $X := \begin{array}{c|c} & x_2 \\ \hline s & 0 \end{array}$ and $M := (\{0,1\})$. Then $M \models_X \neg \phi_f$. Since,

$M \models_X \neg \phi_f \iff (\phi_f, X, 0) \in T_M \iff (\neg\psi, X[M/x_0][F/x_1], 0) \in T_M$ for a $F: X[M/x_0] \to M$.

$\iff (\psi, X[M/x_0][F/x_1], 1) \in T_M$ for a $F: X[M/x_0] \to M$.

The latest can be confirmed by choosing $F(s) = s(x_0)$.

Hence $M \models_X \neg \phi_f$ (1).

On the otherhand, $M \not\models_X \neg\phi$. Since, $M \models_X \neg\phi$

$\iff \ldots \iff (\overset{(*)}{=(x_2, x_1) \wedge x_0 = x_1}, \overbrace{X[M/x_0][F/x_1]}^{X'}, 1) \in T_M$

for some $F: X[M/x_0] \to M$. Assume that this holds, and consider $s(0/x_0), s(1/x_0) \in X[M/x_0]$. By the assumption$^{(*)}$, $F(s(0/x_0)) = 0$ and $F(s(1/x_0)) = 1$.

Hence $\overbrace{s(0/x_0)(0/x_1)}^{t}, \overbrace{s(1/x_0)(1/x_1)}^{t'} \in X'$. But then $t(x_2) = t'(x_2)$ but $t(x_1) \neq t'(x_1)$ which contradicts with $M \models_{X'} =(x_2, x_1)$. Hence we obtain that

$M \not\models_X \neg\phi$. (2). By (1) and (2), $\phi \not\equiv^* \phi_f$. □