Problem Set 2

(1) $\boxed{\phi \vee \top \equiv^* \top}$

(i) For all X, $(\top, X, 1) \in T_M$.

Hence, for all X also there are $Y \cup Z = X$ such that $(Y, \phi, 1) \in T_M$ and $(Z, \top, 1) \in T_M$, since we can choose $Y := \emptyset$ and $Z := X$. We conclude that $M \models_X \phi \vee \top$

$\iff M \models \top$ for all M, X.

(ii) For all X, $(\phi \vee \top, X, 0) \in T_M \iff (\phi, X, 0) \in T_M$ and $(\top, X, 0) \in T_M \iff X = \emptyset \iff (\top, X, 0) \in T_M$.

Hence $M \models_X \neg(\phi \vee \top) \iff M \models_X \neg\top$ for all M, X.

$\boxed{\neg(\phi \wedge \psi) \equiv^* (\neg\phi \vee \neg\psi)}$

(i) $(\neg(\phi \wedge \psi), X, 1) \in T_M \iff (\phi \wedge \psi, X, 0) \in T_M \iff$

$(\phi, Y, 0) \in T_M$ and $(\psi, Z, 0) \in T_M$ for some $Y \cup Z = X$

$\iff (\neg\phi, Y, 1) \in T_M$ and $(\neg\psi, Z, 1) \in T_M$ for some $Y \cup Z = X$

$\iff (\neg\phi \vee \neg\psi, X, 1) \in T_M$.

(ii) $(\neg(\phi \wedge \psi), X, 0) \in T_M \iff (\phi \wedge \psi, X, 1) \in T_M \iff$

$(\phi, X, 1) \in T_M$ and $(\psi, X, 1) \in T_M \iff (\neg\phi, X, 0) \in T_M$

and $(\neg\psi, X, 0) \in T_M \iff (\neg\phi \vee \neg\psi, X, 0) \in T_M$

② $\boxed{\exists x_n \exists x_m \phi \equiv^* \exists x_m \exists x_n \phi}$

(i) We first show that for all $\mathcal{M}, X$, $\mathcal{M} \models_X \exists x_n \exists x_m \phi$

$(\Leftarrow)$ $\mathcal{M} \models_X \exists x_m \exists x_n \phi$. For this, by symmetry, it suffices to show the "$\Rightarrow$" direction only.

So assume that $\mathcal{M} \models_X \overbrace{\exists x_n \exists x_m \phi}^{\psi}$ i.e. $(\psi, X, 1) \in T_{\mathcal{M}}$.

Then there are functions $F : X \to M$ and $G : X[F/x_n] \to M$ such that $(\overbrace{X[F/x_n][G/x_m]}^{X'}, \phi, 1) \in T_{\mathcal{M}}$. By Closure Test, and the truth definition, it suffices to define $G' : X \to M$ and $F' : X[G'/x_m] \to M$ such that $\overbrace{X[G'/x_m][F'/x_n] \subseteq X[F/x_n][G/x_m]}^{(1)}$.

We first define $\overbrace{G'(s) = G(s(F(s)/x_n))}^{(2)}$. Then for each $s \in X[G'/x_m]$ there might be several $s' \in X$ such that $\overbrace{s'(G'(s')/x_m) = s}^{(3)}$. We define $F' : X[G'/x_m] \to M$ as follows: for each $s \in X[G'/x_m]$ we choose one s' of this kind, and define

$$F'(s) = F(s'). \qquad (4)$$

Let us now show (1). Let $t \in X[G'/x_m][F'/x_n]$, $s \in X[G'/x_m]$ such that $F'(s) = t(x_n)$, and $s' \in X$ be the assignment such that $F'(s) = F(s')$. We claim that $s_0 := \overbrace{s'(F(s')/x_n)}^{s''}(G(s'')/x_m) = t$. First note that $s_0(x_n) = F(s') \overset{(4)}{=} F'(s) = t(x_n)$.

(2) Contin

Also $s_0(x_m) \overset{(2)}{=} G(s'') \overset{(3)}{=} G'(s') = s(x_m) = t(x_m)$.

Clearly $s_0 \upharpoonright Dom(X) \setminus \{x_n, x_m\} = t \upharpoonright Dom(X) \setminus \{x_n, x_m\}$. Hence

$s_0 = t$ as wanted.

(ii) $(\exists x_n \exists x_m \phi, X, 0) \in T_n \iff (\phi, X[M/x_n][M/x_m], 0) \in T_n$

$\iff (\phi, \cancel{X}X[M/x_m][M/x_n], 0) \in T_n \iff (\exists x_m \exists x_n \phi, X, 0) \in T_n$.

$\boxed{\neg\forall x_n \phi \equiv \exists x_n \neg \phi}$

(i) $(\neg\forall x_n \phi, X, 1) \in T_n \iff (\forall x_n \phi, X, 0) \in T_n \iff (\phi, X[F/x_n], 0)$

$\in T_n$ for some $F: X \to M \iff (\neg\phi, X[F/x_n], 1) \in T_n$

for some $F: X \to M \iff (\exists x_n \neg\phi, X, 1) \in T_n$.

(ii) $(\neg\forall x_n \phi, X, 0) \in T_n \iff (\forall x_n \phi, X, 1) \in T_n \iff (\phi, X[M/x_n], 1)$

$\in T_n \iff (\neg\phi, X[M/x_n], 0) \in T_n \iff (\exists x_n \neg\phi, X[M/x_n], 0) \in T_n$.

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(3) Let $\phi := {=}(x)$, and let $M := \{0,1\}$ and X be the following team:

	x
s	0
s'	1

. Then $M \models_X \top$ but $M \not\models_X \phi \vee \neg\phi$.

For the latter one $M \models_X \phi \vee \neg\phi$ iff $M \models_Y \phi$ and $M \models_Z \neg\phi$

for some $Y \cup Z = X$. Since $M \models_Z \neg\phi \implies Z = \emptyset$ and $M \not\models_X \phi$,

we obtain that $M \not\models_X \phi \vee \neg\phi$.

Hence $\phi \vee \neg\phi \not\equiv^* \top$.

(4) Let M and X be as in the previous exercise,

and let $\phi := \; =(x)$ again.

Then $M \models_X \phi \vee \phi$ but $M \not\models_X \phi$. Hence $\phi \vee \phi \not\equiv^* \phi$

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(5) $\boxed{\phi \wedge \phi \equiv \phi}$:

$(\phi \wedge \phi, X, 1) \in T_M \iff (\phi, X, 1) \in T_M$ and $(\phi, X, 1) \in T_M$

$\iff (\phi, X, 1) \in T_M$.

$\boxed{\phi \wedge \phi \not\equiv^* \phi}$: Let $\phi := \neg =(x)$, and M and X as defined in exercise (3).

Then $(\phi \wedge \phi, X, 0) \in T_M \iff (\phi, Y, 0) \in T_M$ and $(\phi, Z, 0) \in T_M$ for some $Y \cup Z = X \iff (=(x), Y, 1) \in T_M$ and $(=(x), Z, 1) \in T_M$. for some $Y \cup Z = X$.

Now, choosing $Y := \{s\}$ and $Z := \{s'\}$, we obtain the previous one. Hence, $M \models_X \neg(\phi \wedge \phi)$.

Then again, $(\phi, X, 0) \in T_M \iff (=(x), X, 1) \in T_M$.

However, in this case we clearly have that $(=(x), X, 1) \notin T_M$.

Hence $M \not\models_X \neg\phi$. Therefore, $\phi \wedge \phi \not\equiv^* \phi$.

⑥ Let $\phi := x = y$, $\psi := \neg x = y$, $\theta := =(x)$. Also, let $M := \{0,1\}$ and $X :=$

	x	y
s	0	0
s'	1	0

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Then $M \not\models_X (\phi \wedge \psi) \vee \theta$. Since, for all $Y \neq \emptyset$, we would have that $M \not\models_Y \phi \wedge \psi$, and $M \not\models_X =(x)$.

Then again, $M \models_X (\phi \vee \theta) \wedge (\psi \vee \theta)$. We choose $Y \cup Z = X$ so that $Y := \{s\}$ and $Z := \{s'\}$. Clearly, $M \models_Y \phi \wedge \theta$ and $M \models_Z \psi \wedge \theta$.