Ex 1

(1) We prove a stronger claim by structural induction:

Claim: For a FO formula ϕ,

$$1)\ (\phi^d)^d = (\phi^p)^p = \phi^p$$
$$2)\ (\phi^d)^p = (\phi^p)^d = \phi^d$$
$$3)\ \phi^p \equiv \phi \text{ and } \phi^d \equiv \neg\phi$$

| $\equiv$ denotes logical equivalence.

Proof: Induction on complexity of ϕ.

ATOMIC (i) 1) $(\alpha^d)^d = (\neg\alpha)^d = \alpha^p = (\alpha^p)^p$. | α atomic formula

2) $(\alpha^d)^p = (\neg\alpha)^p = \alpha^d = \neg\alpha = \alpha^d = (\alpha^p)^d$.

3) Clearly, $\alpha^p \equiv \alpha$ and $\alpha^d \equiv \neg\alpha$.

NEGATED (ii) 1) $((\neg\phi)^d)^d = (\phi^p)^d \overset{\text{ind as.}}{=} (\phi^d)^p = ((\neg\phi)^p)^p$.

Moreover, $(\phi^p)^d \overset{\text{ind as.}}{=} \phi^d = (\neg\phi)^p$.

2) $((\neg\phi)^d)^p = (\phi^p)^p \overset{\text{ia}}{=} (\phi^d)^d = ((\neg\phi)^p)^d$.

Moreover $(\phi^p)^p \overset{\text{ia}}{=} \phi^p = (\neg\phi)^d$.

3) Let $\mathcal{M}$ be a model and s an assignment. Then

$\mathcal{M} \models_s (\neg\phi)^p \iff \mathcal{M} \models_s \phi^d \overset{\text{ia}}{\iff} \mathcal{M} \models_s \neg\phi$ and

$\mathcal{M} \models_s (\neg\phi)^d \iff \mathcal{M} \models_s \phi^p \overset{\text{ia}}{\iff} \mathcal{M} \models_s \phi \iff \mathcal{M} \models_s \neg(\neg\phi)$

DISJUNCTION (iii) 1) $((\phi \vee \psi)^d)^d = (\phi^d \wedge \psi^d)^d = (\neg(\neg\phi^d \vee \neg\psi^d))^d = (\neg(\phi^p \vee \psi^p))^d$

$= (\phi^p \vee \psi^p)^p = ((\phi \vee \psi)^p)^p$

Moreover, $(\phi^p \vee \psi^p)^p = (\phi^p)^p \vee (\psi^p)^p \overset{\text{ia}}{=} \phi^p \vee \psi^p = (\phi \vee \psi)^p$.

① 2) $((\phi \vee \psi)^d)^p = (\phi^d \wedge \psi^d)^p = (\neg(\neg\phi^d \vee \neg\psi^d))^p = (\neg(\phi^p \vee \psi^p))^p$

Cont..

$= (\phi^p \vee \psi^p)^d = ((\phi \vee \psi)^p)^d$.

Moreover $(\phi^p \vee \psi^p)^d \stackrel{ia}{=} (\phi \vee \psi)^d$

3) $\mathcal{M} \models_s (\phi \vee \psi)^p \iff \mathcal{M} \models_s \phi^p \vee \psi^p \iff \mathcal{M} \models_s \phi^p$ or $\mathcal{M} \models_s \psi^p$

$\stackrel{ia}{\iff} \mathcal{M} \models_s \phi$ or $\mathcal{M} \models_s \psi \iff \mathcal{M} \models_s \phi \vee \psi$, and

$\mathcal{M} \models_s (\phi \vee \psi)^d \iff \mathcal{M} \models_s \phi^d \wedge \psi^d \iff \mathcal{M} \models_s \phi^d$ and $\mathcal{M} \models_s \psi^d$

$\stackrel{ia}{\iff} \mathcal{M} \models_s \neg\phi$ and $\mathcal{M} \models_s \neg\psi \iff \mathcal{M} \models_s \neg(\phi \vee \psi)$.

EX. QUAN. (iv) 1) $((\exists x \phi)^d)^d = (\forall x \phi^d)^d = (\neg \exists x \neg \phi^d)^d = (\exists x \neg \phi^d)^p$

$= (\exists x \phi^p)^p = ((\exists x \phi)^p)^p$.

Moreover $(\exists x \phi^p)^p = \exists x (\phi^p)^p \stackrel{ia}{=} \exists x \phi^p = (\exists x \phi)^p$.

1 cont...

2) $((\exists x \phi)^d)^p = (\forall x \phi^d)^p = (\neg \exists x \neg \phi^d)^p = (\neg \exists x \phi^p)^p = (\exists x \phi^p)^d$

$= ((\exists x \phi)^p)^d$. Moreover $(\exists x \phi^p)^d = \forall x (\phi^p)^d \overset{ia}{=} \forall x \phi^d = (\exists x \phi)^d$

3) $\mathcal{M} \models_s (\exists x \phi)^p \iff \mathcal{M} \models_s \exists x \phi^p \iff \mathcal{M} \models_{s(a/x)} \phi^p$ for some $a \in M$

$\overset{ia}{\iff} \mathcal{M} \models_{s(a/x)} \phi$ for some $a \in M \iff \mathcal{M} \models_s \exists x \phi$, and

$\mathcal{M} \models_s (\exists x \phi)^d \iff \mathcal{M} \models_s \forall x \phi^d \iff \mathcal{M} \models_{s(a/x)} \phi^d$ for all $a \in M$

$\overset{ia}{\iff} \mathcal{M} \models_{s(a/x)} \neg \phi$ for all $a \in M \iff \mathcal{M} \models_s \forall x \neg \phi \iff \mathcal{M} \models_s \neg \exists x \phi$.

□

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2 NO 1. $\phi := x_0 = x_2$ or $\phi := \neg x_0 = x_2$. In both cases, $\mathcal{M} \models_X \phi$

$\iff (\phi, X, 1) \in T \iff$ for all $s \in X$ $\mathcal{M} \models_s \phi$.

Since $\mathcal{M} \not\models_{s_0} x_0 = x_2$ and $\mathcal{M} \not\models_{s_1} \neg x_0 = x_2$, we obtain that $\mathcal{M} \not\models_X \phi$ in both cases.

YES 2. $\mathcal{M} \models_X \phi \iff \exists F : X \to M$ s.t. $\mathcal{M} \models_{X[F/x_0]} x_0 = x_2$,

where $X[F/x_0] := \{ s(F(s)/x_0) : s \in X \}$. If F is the mapping: $F(s) = 2$ for all $s \in X$, then $X[F/x_0]$ is the following team:

	x_0	x_1	x_2
s	2	2	2
s'	2	1	2

Since $\mathcal{M} \models_s x_0 = x_2$ and $\mathcal{M} \models_{s'} x_0 = x_2$, we obtain that $\mathcal{M} \models_{X[F/x_0]} x_0 = x_2$.

Therefore, $\mathcal{M} \models_X \phi$.

YES 3. $\mathcal{M} \models_X \phi \iff \mathcal{M} \models_{X[M/x_3]} =\!(x_2)$, where $X[M/x_3] := \{ s(a/x_3) :$

$a \in M, s \in X \}$. Moreover, $\mathcal{M} \models_{X[M/x_3]} =\!(x_2)$ if for all $s, s' \in X[M/x_3]$:

$s(\cancel{x_3}) = s'(x_2)$. Clearly, this is the case, hence $\mathcal{M} \models_X \phi$.

② YES 4. $\mathcal{M} \models_X \phi \iff$ for some $Y \cup Z = X$, $\mathcal{M} \models_Y =(x_0, x_1)$

Cont...

and $\mathcal{M} \models_Z =(x_1, x_2)$.

We choose $Y := X$ and $Z := \emptyset$. Then $\mathcal{M} \models_Z =(x_1, x_2)$ by Lemma 3.9 (The Empty Team Property). Moreover, $\mathcal{M} \models_Y =(x_0, x_1)$ $\iff$ for all $s, s' \in Y$: if $s(x_0) = s'(x_0)$, then $s(x_1) = s'(x_1)$.

Clearly the right-hand side holds; hence $\mathcal{M} \models_Y =(x_0, x_1)$ when $\mathcal{M} \models_X \phi$.

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③ We claim: X of $\mathcal{M}$ with domain $\{x_0, x_1\}$ is of type 1 (or 2) exactly when there exists a $f: \mathbb{N} \to \mathbb{N}$ s.t. $s(x_1) = f(s(x_0))$ for all $s \in X$. (i.e. $\mathcal{M} \models_X =(x_0, x_1)$).

Proof:

1. "$\Rightarrow$" Assume that X is of type 1. Then for all $s, s' \in X$: if $s(x_0) = s'(x_0)$, then $s(x_0) + s(x_1) = s'(x_0) + s'(x_1)$. Clearly then, assuming $s(x_0) = s'(x_0)$, it follows that $s(x_1) = s'(x_1)$. Hence $\mathcal{M} \models_X =(x_0, x_1)$.

"$\Leftarrow$" Assume that $\mathcal{M} \models_X =(x_0, x_1)$ when for all $s, s' \in X$: if $s(x_0) = s'(x_0)$, then $s(x_1) = s'(x_1)$. Clearly then for all $s, s' \in X$: if $s(x_0) = s'(x_0)$, then $s(x_0) + s(x_1) = s'(x_0) + s'(x_1)$. Hence $\mathcal{M} \models_X =(x_0, x_0 + x_1)$.

2. "$\Rightarrow$" Assume that X is of type 2 when for all $s, s' \in X$: if $s(x_0) \cdot s(x_0) = s'(x_0) \cdot s'(x_0)$, then $s(x_1) \cdot s(x_1) = s'(x_1) \cdot s'(x_1)$. If now $s, s' \in X$ are such that $s(x_0) = s'(x_0)$, then by the assumption $s(x_1)^2 = s'(x_1)^2$. Since $s(x_1), s'(x_1) \in \mathbb{N}$, we obtain then that $s(x_1) = s'(x_1)$. Hence $\mathcal{M} \models_X =(x_0, x_1)$

③ cont... "$\Leftarrow$" Assume that $\mathcal{M} \models_X =(x_0, x_1)$, and let $s, s' \in X$ be such that $s(x_0)^2 = s'(x_0)^2$. Again, we obtain that $s(x_0) = s'(x_0)$ when by the assumption $s(x_1) = s'(x_1)$ when $s(x_1)^2 = s'(x_1)^2$. Hence $\mathcal{M} \models_X =(x_0 \times x_0, x_1 \times x_1)$. □

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④ Let $\mathcal{M}$ be a model and $X \neq \emptyset$ a team.

It suffices to show that then [1)] $\mathcal{M} \not\models_X \neg\phi$ and [2)] $\mathcal{M} \models_X \phi$:

1) Since $X \neq \emptyset$, we have that $\overbrace{(X, =(x_0, x_1), 0) \notin T}^{\text{by prop. 3.8}}$.

Moreover, by Prop 3.8, $(X, \phi, 0) \notin T$ when $\mathcal{M} \not\models_X \neg\phi$ by Def. 3.5.

2) Let $Y := \{ s \in X : s(x_0) = s(x_1) \}$ and $Z := X \setminus Y$.

Then $\mathcal{M} \models_Y =(x_0, x_1)$ since for all $s, s' \in Y$: if $s(x_0) = s'(x_0)$, then $s(x_1) = s(x_0) = s'(x_0) = s'(x_1)$.

Clearly $\mathcal{M} \models_Z \neg x_0 = x_1$, since for all $s \in Z$: $s(x_0) \neq s(x_1)$.

Hence we obtain that $\mathcal{M} \models_X \phi$, when, for $X \neq \emptyset$,

$$\mathcal{M} \models_X \neg\phi \iff \mathcal{M} \not\models_X \phi.$$ □

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⑤ Let $\mathcal{M}$ be arbitrary model. We show that $\mathcal{M} \models_{\{\emptyset\}} \psi$ where $\psi := \forall x_0 \forall x_1 \overbrace{(x_1 = c \rightarrow =(x_0, x_1))}^{\psi^*}$. That is, we show that $(\{\emptyset\}, \psi, 1) \in T$. Now $(\{\emptyset\}, \psi, 1) \in T$

$\iff (\overbrace{\{\emptyset\}[M/x_0][M/x_1]}^{X}, \psi^*, 1) \in T$. Let $Y := \{ s \in X : s(x_1) \neq c^{\mathcal{M}} \}$

and $Z := X \setminus Y$.

⑤ cont.

Then $(Y, x_1 = c, 0) \in T$. Also $(Z, =(x_0, x_1), 1) \in T$ since for all $s, s' \in Z$: if $s(x_0) = s'(x_0)$, then $s(x_1) = c^M = s'(x_1)$. Hence $(Y, \neg x_1 = c, 1) \in T$ when we obtain that $(X, \neg x_1 = c \vee =(x_0, x_1), 1) \in T$. Note that $\neg x_1 = c \vee =(x_0, x_1)$ $=: x_1 = c \to =(x_0, x_1)$. Hence we have showed that $(X, \psi^*, 1) \in T$ when $(\{\emptyset\}, \psi, 1) \in T$, that is, $\mathcal{M} \models_{\{\emptyset\}} \psi$. Therefore we obtain, since $\mathcal{M}$ is arbitrary, that $\models \psi$. $\square$

(ordering of triples $(\phi, X, 1), \ldots$ is not here as it is in the book...)

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⑥ Induction claim would be:

"If $Y \subseteq X$, then $(\phi, X, 1) \in T \Rightarrow (\phi, Y, 1) \in T$

and $(\phi, X, 0) \in T \Rightarrow (\phi, Y, 0) \in T$."

NEGATION (i) $\phi = \neg\psi$. Then $(\phi, X, 1) \in T \Leftrightarrow (\psi, X, 0) \in T \overset{\text{ia}}{\Rightarrow} (\psi, Y, 0) \in T$

$\Leftrightarrow (\phi, Y, 1) \in T$, and

$(\phi, X, 0) \in T \Leftrightarrow (\psi, X, 1) \in T \overset{\text{ia}}{\Rightarrow} (\psi, Y, 1) \in T \Leftrightarrow (\phi, Y, 0) \in T$.

EXISTENTIAL (ii) $\phi = \exists x_n \psi$. Then $(\phi, X, 1) \in T \Leftrightarrow (\psi, X[F/x_n], 1) \in T$ for some $F: X \to M$

$\overset{\text{ia}}{\Rightarrow} (\psi, Y[G/x_n], 1) \in T$ for some $G: Y \to M$ $\Leftrightarrow (\phi, Y, 1) \in T$,

and $(\phi, X, 0) \in T \Leftrightarrow (\psi, X[M/x_n], 0) \in T \overset{\text{ia}}{\Rightarrow} (\psi, Y[M/x_n], 0) \in T$

$\Leftrightarrow (\phi, Y, 0) \in T$. $\square$