Exercise Set 4

If you need credit for the course, please turn in half of the following exercises (of a choice of your own).

1. We have that $A = \mathbf{k}[SL_2] = \mathbf{k}[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3 - 1)$. Let t_i denote the image of T_i in A and B the subalgebra of A generated by t_it_j, $i, j = 1, ..., 4$.

(a) Show that B is a Hopf algebra. Let PSL_2 be the linear algebraic group whose algebra is B. Show that the inclusion map $B \to A$ defines a homomorphism of algebraic groups $\phi : SL_2 \to PSL_2$ with kernel of order at most two.
(b) If $\operatorname{char} \mathbf{k} \neq 2$, then B is the algebra of functions $f \in A$ such that $f(-X) = f(X)$ for all $X \in SL_2$.
(c) If $\operatorname{char} \mathbf{k} = 2$, ϕ defines an isomorphism of abstract groups but is not an isomorphism of algebraic groups.
(d) Show that $d\phi$ is bijective if and only of $p \neq 2$. Describe $d\phi$ when $p = 2$.

2. Let G be a linear algebraic group and H a closed subgroup. An open subvariety of an affine algebraic variety is called a quasi-affine variety. If all the homomorphisms of linear algebraic groups $H \to \mathbb{G}_m$ are trivial, then G/H is quasi-affine.

3. Let $G = SL_2$.

(a) If H is the subgroup of upper triangular matrices then G/H is isomorphic to the projective line $\mathbb{P}^1$.
(b) If H is the subgroup of upper triangular unipotent matrices, then G/H is isomorphic to $\mathbb{A}^2 - \{0\}$ (a quasi-affine variety that is not affine, why?).

4. Let G be a linear algebraic group and H a closed subgroup. If G/H is affine, then

$$\mathbf{k}[G/H] = A := \{f \in \mathbf{k}[G] \mid f(gh) = f(g) \text{ for all } g \in G, h \in H\}.$$

5. Let G be a connected linear algebraic group and B a Borel subgroup of G. If σ is an automorphism of the algebraic group G that fixes all elements of B, then $\sigma = \mathrm{id}$.

6. Give an example of a finite solvable subgroup of $SL_2(\mathbb{C})$ that is not conjugate to a group of upper triangular matrices.

7. Let G be a connected linear algebraic group. Call $x \in G$ regular if the multiplicity of the root 1 of the characteristic polynomial of the linear map $\operatorname{Ad} x$ of $\mathfrak{g}$ is minimal.

(a) the regular elements form a non-empty open subset of G.
(b) $x \in G$ is regular if and only if its semi-simple part x_s is regular.

(c) A semi-simple element is regular if and only if its centralizer has minimal dimension.
(d) A semi-simple element x is regular if an only if $Z_G(x)^0$ is a Cartan subgroup.

8. A maximal nilpotent, closed subgroup C of a connected linear algebraic group G such that $C = N_G(C)^0$ is a Cartan subgroup.

9. Describe the root datum for the groups GL_n, SL_n, $PGL_n = GL_n/C(GL_n)$, SO_{2n+1} (char $\mathbf{k} \neq 2$), Sp_{2n} (char $\mathbf{k} \neq 2$).

Show that the root datum of SL_n is dual to that of PGL_n, and the root datum of Sp_{2n} is dual to that of SO_{2n+1}.

Here SO_{2n+1} is the subgroup of SL_{2n+1} consisting of elements that preserves the quadratic form

$$Q((v_1, \ldots, v_{2n+1})) = \sum_{i=1}^{n} v_i v_{n+i} + v_{2n+1}^2,$$

and Sp_{2n} is the subgroup of GL_{2n} consisting of elements that preserves the symplectic form

$$((v_1, \ldots, v_{2n}), (w_1, \ldots, w_{2n})) = \sum_{i=1}^{n} v_i w_{n+i} - v_{n+i} w_i.$$

(*It would be nice if one can at least work out the root system.*)