Exercise Set 3

If you need credit for the course, please turn in half of the following exercises (of a choice of your own).

1. Give an example of a linear algebraic group which consists of semisimple elements but is not diagonalizable.

2. Let $\phi : G \to H$ be a homomorphism of *diagonalizable* groups. Denote by ϕ^* the induced homomorphism $X^*(H) \to X^*(G)$. Show that if ϕ is injective (resp. surjective) then ϕ^* is surjective (resp. injective).

3. The group of automorphisms of an n-dimensional torus is isomorphic to the group $GL_n(\mathbb{Z})$ of integral $n \times n$-matrices with an integral inverse.

4. Using $T_xX \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, describe T_xX in the following cases
   (a) X is a point,
   (b) $X = \mathbf{A}^n$,
   (c) $X = \{(a,b) \in \mathbf{A}^2 \mid ab = 0\}$, $x = (0,0)$,
   (d) ($\operatorname{char} \mathbf{k} \neq 2, 3$) $X = \{(a,b) \in \mathbf{A}^2 \mid a^2 = b^3\}$, $x = (0,0)$.

5. Let X and Y be algebraic varieties, $x \in X, y \in Y$. Show that $T_{x,y}(X \times Y) \simeq T_xX \oplus T_yY$.

6. If X is a closed subvariety of Y and $\phi : X \to Y$ is the injection morphism, then $d\phi_x$ is injective for all $x \in X$.

7. (a) If $A = R[T_1, \dots, T_m]$, then $\Omega_{A/R}$ is a free A-module with basis $(dT_i)_{1 \le i \le m}$.
   (b) Let A be an R-algebra which is an integral domain and let F be the quotient filed of A. Then $\Omega_{F/R} \cong F \otimes_A \Omega_{A/R}$.
   (c) Let F be a field and $E = F(x_1, \dots, x_m)$ be an extension field of finite type. Then $\Omega_{E/F}$ is a finite dimensional vector space over E spanned by dx_i.
   (d) Let $A = \mathbf{k}[T,U]/(T^2 - U^3)$. Show that $\Omega_{A/\mathbf{k}}$ is not a free A-module.
   (e) Let A and B be R-algebras. There is an isomorphism of $A \otimes_R B$-modules
   $$\Omega_{A \otimes_R B/R} \simeq (\Omega_{A/R} \otimes_R B) \oplus (A \otimes_R \Omega_{B/R})$$
   under which $d_{A \otimes_R B/R}$ corresponds to $(d_{A/R} \otimes_R \mathrm{id}_B) \oplus (\mathrm{id}_A \otimes_R d_{B/R})$.

8. (a) Let H be a closed subgroup of a linear algebraic group G and let $J \subset \mathbf{k}[G]$ be the ideal of functions vanishing on H so that $\mathbf{k}[H] = \mathbf{k}[G]/J$. Using the fact that $\operatorname{Lie} H = \operatorname{Lie} G \cap \{D \in \mathcal{D}_G \mid DJ \subset J\}$ to show that the Lie algebra of SL_n is the subalgebra $\mathfrak{sl}_n$ of $\mathfrak{gl}_n$ of traceless matrices.
   (b) Determine the Lie algebras of $\mathbf{D}_n$ (diagonal), $\mathbf{T}_n$ (upper triangular), $\mathbf{U}_n$ (unipotent upper triangular).
   (c) Let T be a torus. There is a canonical isomorphism $\operatorname{Lie} T \to \mathbf{k} \otimes_{\mathbb{Z}} X_*(T)$.
   (d) $\operatorname{Lie} G = \operatorname{Lie} G^0$.

9. The differential of the adjoint representation Ad is given by
$$d\mathrm{Ad}(X)(Y) = [X, Y]. \quad \text{(Hint: first prove for } GL_n.)$$