Exercise Set 2

If you need credit for the course, please turn in half of the following exercises (of a choice of your own).

1. (a) Show that the groups $\mathbb{G}_a$, $\mathbb{G}_m$, GL_n, $\mathbf{D}_n$ (diagonal matrices), $\mathbf{U}_n$ (upper triangular matrices), SL_n are connected.
   (b) Assume that the characteristic of $\mathbf{k}$ is not 2. Show that $O_n = \{g \in GL_n \,|\, {}^t g.g = I\}$ is not connected.

2. (a) The only automorphisms of $\mathbb{G}_m$ (as linear algebraic groups) are $x \mapsto x$ and $x \mapsto x^{-1}$.
   (b) $\operatorname{Aut} \mathbb{G}_a \cong \mathbf{k}^*$.

3. Show by an example that the subgroup of an algebraic group generated by two non-irreducible closed subsets need not be closed. (Consider the cyclic subgroups of $GL_2(\mathbb{C})$ generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.)

4. Let G be a connected algebraic group. Prove that any finite normal subgroup H lies in the center $Z(G) = \{g \in G \,|\, gh = hg \text{ for all } g, h \in G\}$.

5. Let G be a closed subgroup of GL_n. Then $\mathbf{A}^n$ has a natural G-action. Determine the obits for $GL_n, \mathbf{D}_n$, SL_n.

6. Let G be a subgroup of GL_n which acts irreducibly in $V = \mathbf{k}^n$. Show that the only normal unipotent subgroup of G is the trivial one.

7. Show that $\mathbf{k}[G]$ is the union of finite dimensional subspaces stable under $\rho(G)$.

8. Let $a \in GL(V)$, $b \in GL(W)$. Show that $(a \otimes b)_s = a_s \otimes b_s$, $(a \otimes b)_u = a_u \otimes b_u$.

9. Assume that the characteristic of $\mathbf{k}$ is zero.
   (a) An element of GL_n having finite order must be semisimple.
   (b) An element of $GL(V)$ is unipotent if and only if there exists a rational representation $\mathbb{G}_a \to GL(V)$ whose image contains x.

10. (a) Show that the set G_u of unipotent elements of G is closed.
    (b) Show by an example that the set G_s of semi-simple elements of G is not necessarily open or closed.